Math Database
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11/9/2023

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Introduction

The approach for this writeup is to shift from performing the computations numerically or symbolically by hand, to automating the computations symbolically using Python and the sympy library. The idea is to automate and speed up the process of performing the computations while retaining their generality. The math database will be able to store results from calculations, so they don't need to be repeated and can be reused in more complex calculations. The code associated with this writeup is located at MathDatabase.

Format

In general, coordinates such as polar don't depend on any other coordinates. For example, you could create a grid in polar coordinates and plot polar point tuples $(r \quad \theta)$ without referencing cartesian coordinates at all. However, in that case you can't add vectors by adding their components algebraically as will be shown. When defining curvilinear coordinates, as we have in the past, there are a set of 'new' coordinates written as linear combinations of a set of 'old' coordinates. But to create new vectors by writing them as linear combinations of a set of 'old' coordinates, means that the 'old' basis vector components need to be able to be added algebraically and implies that all curvilinear coordinate systems were derived from cartesian components and that the basis vectors of these coordinate systems are written as cartesian representations. For example, polar coordinates will be derived from cartesian coordinates, so the polar basis vectors are written in terms of cartesian components. A new coordinate system derived from a linear combination of the polar basis vectors, will therefore also be written in terms of cartesian components.

As we have seen in previous writeups, the relationships between the old and new coordinates is given by equation (1).

$$q^{i}(q^{\bar{j}}) \Rightarrow \boldsymbol{q}(\overline{\boldsymbol{q}}) \equiv \text{coordinate relation} \tag{1}$$

Note: that the primed coordinates are notated by putting a bar above the index. This notation allows the transformation matrices to be notated in a functional form as functions of the indices. The transformation of basis vectors from 'old' to 'new coordinates, E to $\bar{E}$ is given by equation (2).

$$\frac{\partial}{\partial q^{\bar{1}}} = \frac{\partial q^1}{\partial q^{\bar{1}}}\frac{\partial}{\partial q^1} + \cdots + \frac{\partial q^n}{\partial q^{\bar{1}}}\frac{\partial}{\partial q^n}$$

$$\vdots$$

$$\frac{\partial}{\partial q^{\bar{n}}} = \frac{\partial q^1}{\partial q^{\bar{n}}}\frac{\partial}{\partial q^1} + \cdots + \frac{\partial q^n}{\partial q^{\bar{n}}}\frac{\partial}{\partial q^n} \tag{2}$$

The A matrix is given by equation (3).

$$A = \frac{\partial q^j}{\partial q^{\bar{\imath}}} = \begin{bmatrix} \frac{\partial q^1}{\partial q^{\bar{1}}} & \cdots & \frac{\partial q^n}{\partial q^{\bar{1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial q^1}{\partial q^{\bar{n}}} & \cdots & \frac{\partial q^1}{\partial q^{\bar{n}}} \end{bmatrix} = A_{\bar{\imath}j} = A(\underset{\text{row}}{\bar{\imath}} \quad \underset{\text{column}}{j}) = A(new \quad old) \tag{3}$$

In equation (3), the unprimed indices index the rows and the primed indices index the columns of the A matrix.

Equation (2) in matrix form is given by equation (4)

$$\bar{E} = AE \tag{4}$$

where

$$\boldsymbol{e_i} = E = \frac{\partial}{\partial q^i} \equiv basis\ vectors\ in\ q^i\ coordinates \equiv old\ coordinates$$

$$\boldsymbol{e_{\bar{\imath}}} = \bar{E} = \frac{\partial}{\partial q^{\bar{\imath}}} \equiv basis\ vectors\ in\ q^{\bar{\imath}}\ coordinates \equiv new\ coordinates$$

Notice that equation (4) shows that the $\bar{E}$ matrix gives the new coordinate bases in terms of the old bases.

Now we show how to do multiple transformations. First, subscript the matrices in equation (4).

$$E_2 = A_1 E_1 \tag{5}$$

Now perform another transformation using $A_2 \Rightarrow$

$$E_3 = A_2 E_2 = A_2 A_1 E_1 \tag{6}$$

In general,

$$E_N = A_{N-1}E_{N-1} = A_{N-1}A_{N-2}\cdots A_1E_1 \tag{7}$$

Equation (8) is the inverse coordinate relation.

$$q^{\bar{\imath}}\left(q^j\right) \Rightarrow \bar{\boldsymbol{q}}(\boldsymbol{q}) \equiv \text{inverse coordinate relation} \tag{8}$$

The following are the transformation equations for the one form bases:

$$dq^{\bar{1}} = \frac{\partial q^{\bar{1}}}{\partial q^1}dq^1 + \cdots + \frac{\partial q^{\bar{1}}}{\partial q^n}dq^n$$

$$\vdots$$

$$dq^{\bar{n}} = \frac{\partial q^{\bar{n}}}{\partial q^1}dq^1 + \cdots + \frac{\partial q^{\bar{n}}}{\partial q^n}dq^n \tag{9}$$

Let

$$B = \frac{\partial q^{\bar{\jmath}}}{\partial q^i} = \frac{\partial \bar{\boldsymbol{q}}}{\partial \boldsymbol{q}} = \begin{bmatrix} \dfrac{\partial q^{\bar{1}}}{\partial q^1} & \cdots & \dfrac{\partial q^{\bar{n}}}{\partial q^1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial q^{\bar{1}}}{\partial q^n} & \cdots & \dfrac{\partial q^{\bar{1}}}{\partial q^n} \end{bmatrix} = B(\underset{\substack{\uparrow \\ \text{row}}}{i} \quad \underset{\substack{\uparrow \\ \text{column}}}{\bar{\jmath}}) = B(old \quad new) \tag{10}$$

The transformation of basis one forms from ‘old’ to ‘new coordinates, W to $\bar{W}$ is given by equation (11).

$$\bar{W} = B^TW \;\Rightarrow \tag{11}$$

Now we show how to do multiple transformations of the one form bases. Using the same process as we did with basis vectors, subscript the matrices in equation (11).

$$W_2 = [B_1]^TE_1 \tag{12}$$

Now do another transformation using $B_2 \;\Rightarrow$

$$W_3 = [B_2]^TE_2 = [B_2]^T[B_1]^TW_1 \tag{13}$$

In general,

$$W_N = [B_{N-1}]^T W_{N-1} = [B_{N-1}]^T [B_{N-2}]^T \cdots [B_1]^T W_1 \tag{14}$$

Equations (4) and (11) show that both the vector and one form bases transformations are given in terms of old and new coordinates, so we can store the matrices in a record as functions of the tuple $(new \quad old)$. Remember, the bases are row vectors.

Table 1 shows that the transform matrices can be stored in the database using the following format:

Variable Name	Variable Functional Form
A	$A(new \quad old)$
B	$B(old \quad new)$

Table 1

Both matrices will be stored in a record that will be accessed by the tuple $(new \quad old)$. As stated before, all the coordinate systems will be derived from cartesian coordinates, so 'old' will always be written in a cartesian representation.

Vector Addition of Polar Basis Vectors using both Cartesian and Polar Parameters.

To better understand the cartesian representation of basis vectors, we will go through an example using polar basis vectors. Equation (15) shows the relationships to convert from cartesian to polar coordinates.

$$x = r\, cos(\theta)$$
$$y = r\, sin(\theta) \tag{15}$$

Now compute the polar basis vectors. Equation (2) ⇒

$$e_r = \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = cos(\theta)\frac{\partial}{\partial x} + sin(\theta)\frac{\partial}{\partial y} = cos(\theta)e_x + sin(\theta)e_y$$

$$e_\theta = \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -rsin(\theta)\frac{\partial}{\partial x} + rcos(\theta)\frac{\partial}{\partial y} = -rsin(\theta)e_x + rcos(\theta)e_y \tag{16}$$

It's clear that e_r and e_θ are in cartesian coordinates.

Now represent the basis vectors in terms of polar parameters.

Let

$$r = 2$$
$$\theta = \frac{\pi}{3}$$

$$e_r(x, y) = \left[cos\left(\frac{\pi}{3}\right) \quad sin\left(\frac{\pi}{3}\right)\right] = \left[\frac{1}{2} \quad \frac{\sqrt{3}}{2}\right]$$

$$e_\theta(x, y) = \left[-2sin\left(\frac{\pi}{3}\right) \quad 2cos\left(\frac{\pi}{3}\right)\right] = [-\sqrt{3} \quad 1] \tag{17}$$

Now convert the basis vectors to use the parameters to $(r \quad \theta)$ using the inverse coordinate relationship.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) \tag{18}$$

For $e_r(x, y) \Rightarrow$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = atan2\left(\frac{\sqrt{3}}{2} \quad \frac{1}{2}\right) = \frac{\pi}{3} \Rightarrow$$

$$e_r(r, \theta) = \left[1 \quad \frac{\pi}{3}\right] \tag{19}$$

$e_\theta(x, y) \Rightarrow$

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2} = 2$$

$$\theta = atan2(1 \quad -\sqrt{3}) = \frac{5\pi}{6} \Rightarrow \tag{20}$$

$$e_\theta(r,\theta) = \begin{bmatrix} 2 & \frac{5\pi}{6} \end{bmatrix}$$

(21)

Now create a new basis vector that is a linear combination of $e_r(x,y)$ and $e_\theta(x,y)$ ⇒

$$e_{\bar{r}}(x,y) = e_r(x,y) + e_\theta(x,y) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \sqrt{3} & \frac{\sqrt{3}}{2} + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-2\sqrt{3}}{2} & \frac{\sqrt{3}+2}{2} \end{bmatrix}$$

(22)

$e_{\bar{r}}(x,y)$ ⇒

$$r = \sqrt{\left(\frac{1}{2} - \sqrt{3}\right)^2 + \left(\frac{\sqrt{3}}{2} + 1\right)^2} = \sqrt{\left(\frac{1-2\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}+2}{2}\right)^2}$$

Simplifying each component separately ⇒

$$\left(\frac{1-2\sqrt{3}}{2}\right)^2 = \frac{\left(1-2\sqrt{3}\right)\left(1-2\sqrt{3}\right)}{4} = \frac{1-4\sqrt{3}+12}{4} = \frac{13-4\sqrt{3}}{4}$$

$$\left(\frac{\sqrt{3}+2}{2}\right)^2 = \frac{\left(\sqrt{3}+2\right)\left(\sqrt{3}+2\right)}{4} = \frac{3+4\sqrt{3}+4}{4} = \frac{7+4\sqrt{3}}{4} \Rightarrow$$

$$\left(\frac{1-2\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}+2}{2}\right)^2 = \frac{13-4\sqrt{3}}{4} + \frac{7+4\sqrt{3}}{4} = \frac{20}{4} = 5 \Rightarrow$$

$$r = \sqrt{5}$$

(23)

$$\theta = tan^{-1}\left(\frac{\sqrt{3}+2}{1-2\sqrt{3}}\right) = 2.154346268990688$$

(24)

$$e_{\bar{r}}(r \quad \theta) = \begin{bmatrix} \sqrt{5} & 2.154346268990688 \end{bmatrix}$$

(25)

Now add the polar basis vectors in the $(r \quad \theta)$ parameter form directly without using cartesian coordinates. Vectors in polar parameters add according to the following formulas[1]

For

$$v_1(r \quad \theta) = [r_1 \quad \theta_1]$$

$$v_2(r \quad \theta) = [r_2 \quad \theta_2]$$

$$v_1(r \quad \theta) + v_2(r \quad \theta) \Rightarrow$$

$$r = \sqrt{r_1^2 + r_2^2 + 2r_1r_2cos(\theta_1 - \theta_2)} \tag{26}$$

$$\theta = \theta_1 + atan2(r_2sin(\theta_2 - \theta_1) \quad r_1 + r_2cos(\theta_2 - \theta_1)) \tag{27}$$

$$e_r(r,\theta) = \begin{bmatrix} 1 & \frac{\pi}{3} \end{bmatrix}$$

$$e_\theta(r,\theta) = \begin{bmatrix} 2 & \frac{5\pi}{6} \end{bmatrix}$$

$$e_{\bar{r}}(r \quad \theta) = e_r(r,\theta) + e_\theta(r,\theta) \Rightarrow$$

$$r = \sqrt{1^2 + 2^2 + 2(1)(2)cos\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)} = \sqrt{5 + 4cos\left(\frac{\pi}{2}\right)} = \sqrt{5} \tag{28}$$

$$\theta = \theta_1 + atan2(r_2sin(\theta_2 - \theta_1) \quad r_1 + r_2cos(\theta_2 - \theta_1))$$

$$\theta_2 - \theta_1 = \frac{\pi}{2} \Rightarrow$$

$$sin\left(\frac{\pi}{2}\right) = 1$$

$$cos\left(\frac{\pi}{2}\right) = 0 \Rightarrow$$

$$r_2sin(\theta_2 - \theta_1) = r_2 = 2$$

$$r_1 + r_2cos(\theta_2 - \theta_1) = r_1 = 1$$

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[1] Adding two Polar Vectors

$$\theta = \theta_1 + atan2(2 \quad 1) = \frac{\pi}{3} + atan2(2 \quad 1) = 2.154346268990688 \tag{29}$$

Using equations (28) and (29) ⇒

$$e_{\bar{r}}(r \quad \theta) = [\sqrt{5} \quad 2.154346268990688] \tag{30}$$

Equation (30) is the same as equation (25). This discussion shows that curvilinear transformation equations for basis vectors – equation (2) – implicitly assume the polar basis vectors are in cartesian coordinates because the vector components are algebraically added. The polar basis vectors add algebraically in cartesian coordinates but add using more complicated expressions in polar coordinates. Curvilinear coordinates assume that vector components add algebraically, so they represent coordinate bases of a general coordinate system in cartesian coordinates. For this reason, the 'old' coordinates will always be derived from cartesian coordinates. For example, if $'old' = 'polar'$, then it is assumed that $'polar'$ was derived from cartesian coordinates ⇒ the polar basis vectors are in cartesian coordinates and will be shown in later examples.

Database System Outline

The data in the system is processed using the following classes. The classes are written in python and the sympy library is used for computations.

1.) mathDB class
2.) Various coordinate transformation classes
3.) computeMatrices class
4.) transformRecord class
5.) coordinateRecord class

1.) mathDB

The mathDB class contains the _dictionary attribute which is the main attribute that holds the various tables. Currently, there are two tables whose attributes are _transformTable and _coordinateTable. Both of these tables are python dictionaries. The following code, in Figure 1, gets the _transformTable and _coordinateTable from the _dictionary.

```
self._transformTable = self._dictionary['transformTable']
self._coordinateTable = self._dictionary['coordinateTable']
```

Figure 1

Records are retrieved from the transform table and coordinate table using the tuple key $(new \quad old)$.

where

$old \equiv$ old coordinates string $\equiv$ always derived from cartesian coordinates
$new \equiv$ new coordinates string

The user can specify a string representing the coordinate system. For example, Cartesian to polar tuple could be $('polar' \quad 'cartesian')$

More tables can be added to the dictionary as desired.
An example of how the tables are accessed is given in Figure 2 below.

```
polarFromCartesianRecord = transformTable[('polar', 'cartesian')]
```

Figure 2

The _dictionary is saved to a file and loaded from a file using pickle. This file is specified in the call to the class in the __init__ method and the file is read using the pickle.load command. There is a close method in the class that writes the _dictionary to the file using the pickle.dump command.

2.) Various Coordinate Transform Classes

Coordinate classes define the coordinate transformations from an unprimed to a primed coordinate system in vector form. For example, the coordinate relationship of cartesian in terms of polar coordinates is shown in equation (15).

$$x = r\,cos(\theta)$$
$$y = r\,sin(\theta) \tag{15}$$

where

$r > 0$
$0 \leq \theta \leq 2\pi$

Equation (15) can be represented in the abstract form of equation (1) as shown in equation (31).

$$q^i(q^{\bar{j}}) \Rightarrow \boldsymbol{q}(\bar{\boldsymbol{q}}) = [x(r,\theta) \quad y(r,\theta)] = [r\,cos(\theta) \quad r\,cos(\theta)] \tag{31}$$

There are two variable arrays in equation (31) that are shown in equation (32).

$$[r \quad \theta]$$

$$[r\,cos(\theta) \quad r\,sin(\theta)]$$

(32)

Each coordinate transformation is a class and has two attributes - _params and _vec to represent equation (1) and to calculate the A matrix.

In terms of equation (31) ⇒

$$\_params = \boldsymbol{\bar{q}}$$

$$\_vec = \boldsymbol{q}(\boldsymbol{\bar{q}})$$

The _params attribute for equation (31) is $[r \quad \theta]$

The _vec attribute for equation (32) is $[r\,cos(\theta) \quad r\,sin(\theta)]$ ⇒

To compute the B matrix, use the inverse coordinate relation defined in equation (6).

$q^{\bar{\imath}}\left(q^{j}\right) \Rightarrow \boldsymbol{\bar{q}}(\boldsymbol{q}) \equiv$ inverse coordinate relation

(6)

The inverse relationship of transforming cartesian to polar coordinates is shown in equation (33).

$$[r(x,y) \quad \theta(x,y)] = \left[\sqrt{x^2+y^2} \quad tan^{-1}\left(\frac{y}{x}\right)\right]$$

(33)

We add two attributes to the coordinate classes - _inv_params and _inv_vec ⇒

$$\_inv\_params = \boldsymbol{q} = [x \quad y]$$

$$\_inv\_vec = \boldsymbol{\bar{q}}(\boldsymbol{q}) = [r(x,y) \quad \theta(x,y)] = \left[\sqrt{x^2+y^2} \quad tan^{-1}\left(\frac{y}{x}\right)\right]$$

Figure 3 shows the class to transform from Cartesian to Polar

```
class polarFromCartesian:

    def __init__(self, _r, _theta, _x, _y):
        self._name = 'polarFromCartesian'
        self._params = sp.Array([_r, _theta])
        self._vec = sp.Array([self._params[0]*sp.cos(self._params[1]), self._params[0]*sp.sin(self._params[1])]) # definitions of x and y
        self._inv_params = sp.Array([_x, _y])
        self._inv_vec = sp.Array([sp.sqrt(self._inv_params[0]**2 + self._inv_params[1]**2), sp.atan2(_y, _x)])
```

Figure 3

3.) computeMatrices

The computeMatrices class computes the transform matrices – A and B. The A matrix is defined by equation (34).

$$A = \frac{\partial q^j}{\partial q^{\bar{\imath}}} = \frac{\partial \boldsymbol{q}}{\partial \bar{\boldsymbol{q}}} = A_{\bar{\imath}j} = A(\bar{\imath} \quad j) = \begin{bmatrix} \frac{\partial q^1}{\partial q^{\bar{1}}} & \cdots & \frac{\partial q^n}{\partial q^{\bar{1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial q^1}{\partial q^{\bar{n}}} & \cdots & \frac{\partial q^1}{\partial q^{\bar{n}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{q}}{\partial q^{\bar{1}}} \\ \vdots \\ \frac{\partial \boldsymbol{q}}{\partial q^{\bar{n}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial(\_vec)}{\partial\_param_1} \\ \vdots \\ \frac{\partial(\_vec)}{\partial\_param_n} \end{bmatrix} \tag{34}$$

Notice in equation (34) that the n^{th} row of the A matrix is the derivative of the _vec parameter with respect to $\_param_n = \_param[n]$.

The B matrix is defined by equation (35).

$$B = \frac{\partial q^{\bar{\jmath}}}{\partial q^i} = \frac{\partial \bar{\boldsymbol{q}}}{\partial \boldsymbol{q}} = \begin{bmatrix} \frac{\partial q^{\bar{1}}}{\partial q^1} & \cdots & \frac{\partial q^{\bar{n}}}{\partial q^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial q^{\bar{1}}}{\partial q^n} & \cdots & \frac{\partial q^{\bar{1}}}{\partial q^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{\boldsymbol{q}}}{\partial q^1} \\ \vdots \\ \frac{\partial \bar{\boldsymbol{q}}}{\partial q^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial(\_inv\_vec)}{\partial\_inv\_param_1} \\ \vdots \\ \frac{\partial(\_inv\_vec)}{\partial\_inv\_param_n} \end{bmatrix} \tag{35}$$

Again, notice in equation (35) that the n^{th} row of the A matrix is the derivative of the _vec parameter with respect to $\_param_n = \_param[n]$.

Because the computing pattern is the same for both the A and B matrices, we use a single routine to compute both. This routine is shown in Figure 4.

```
def computeTransform(self, _params, _vec):

    _n = len(_params)
    ret = sp.Matrix(0, 0, [])

    for i in range(_n):
        ret = ret.row_insert(i, sp.Matrix([sp.diff(_vec, _params[i])]))

    return ret
```

Figure 4

To compute the A matrix perform the following call to computeTransform ⇒

```
ret = self.computeTransform(params, vec)
```

To compute the B matrix perform the following call to computeTransform ⇒

```
ret = self.computeTransform(inv_params, inv_vec)
```

The computeMatrices class, computes the A and B matrices symbolically using sympy. When the B matrix is computed using equation (35), we need to make a substitution to get the same parameters as are in the A matrix. Having the same parameters makes it possible to get the same result as we do by inverting the A matrix.

As an example, consider the inverse transform - polar in terms of cartesian coordinates.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) \tag{18}$$

$$B = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & -\frac{y}{x^2 + y^2} \\ \frac{y}{\sqrt{x^2 + y^2}} & \frac{x}{x^2 + y^2} \end{bmatrix} \tag{36}$$

Equation (36) is the B matrix in rectangular components. The next step is to use a substitution to convert equation (36) to polar coordinates.

Substitutions in sympy can be done using a dictionary. Equation (15) is the forward transform equation – cartesian in terms of polar.

$$x = r\,cos(\theta)$$
$$y = r\,sin(\theta) \tag{15}$$

Equation (15) gives the substitutions we want to use. Now set up a dictionary as follows:

$$substr = \{x: r\,cos(\theta), y: r\,sin(\theta)\}$$

Replace x by $r\,cos(\theta)$, replace y by $r\,sin(\theta)$.

The subs function in sympy performs the substitution. The code below substitutes and simplifies the B matrix ⇒

$B = B.subs(substr)$

$B = sympy.simplify(B)$

Equation (37) shows the B matrix after the substitution and simplification.

$$B(polar, cartesian) = \begin{bmatrix} \frac{r\cos(\theta)}{\sqrt{r^2}} & -\frac{\sin(\theta)}{r} \\ \frac{r\sin(\theta)}{\sqrt{r^2}} & \frac{\cos(\theta)}{r} \end{bmatrix} \tag{37}$$

Sympy is not simplifying $\sqrt{r^2} \to r$

We add the substitution for all the parameters $\Rightarrow$ $\sqrt{r^2} = r$ and $\sqrt{\theta^2} = \theta$. The code snippet to do this substitution is shown below in Figure 5.

```
# handle sqrt(r**2) = r
params = _coords._params
sub_str = map(lambda x: sp.sqrt(x**2), params)
sub_str = dict(zip(sub_str, params))
ret = ret.subs(sub_str)
ret = sp.simplify(ret)
```

Figure 5

Equation (38) shows the resulting B matrix and is the same matrix that we get from inverting the A matrix.

$$B = \begin{bmatrix} \cos(\theta) & -\frac{\sin(\theta)}{r} \\ \sin(\theta) & \frac{\cos(\theta)}{r} \end{bmatrix} \tag{38}$$

The $\bar{E}$ and $\bar{W}$ matrices are computed using equations (39).

$\bar{E} = AE$

$\bar{W} = B^T W$

(39)

The matrices $A, B, \bar{E}, and\ \bar{W}$ give all the variables necessary to fill Table 1. Note: $\bar{E}$ in the old system becomes E in the new coordinate system which is why the transform record uses the names E and W as opposed to $\bar{E}$ and$\bar{W}$.

4.) transformRecord

The transform record is shown in Figure 6.

```
class transformRecord:

    def __init__(self, _A, _B, _E, _W):
        self._A = _A
        self._B = _B
        self._E = _E
        self._W = _W

    def printRecord(self, _key):
        latex = convertToLatex()

        A_latex = latex.convertMatrixToLatex(self._A)
        print('A',str(_key),' = \n', A_latex, '\n')
        E_latex = latex.convertMatrixToLatex(self._E)
        print('E',str(_key),' = \n', E_latex, '\n')
        B_latex = latex.convertMatrixToLatex(self._B)
        print('B',str(_key),' = \n', B_latex, '\n')
        W_latex = latex.convertMatrixToLatex(self._W)
        print('W',str(_key),' = ', W_latex, '\n\n')
```

Figure 6

Note: the matrices are sympy Matrix types. Also, there is a printRecord method to output the matrices in MS Word latex compatible format that can be cut and pasted into Word's equation editor in latex mode. Note that in the cases of more complex equations, there can be some significant rendering time.

5.) **coordinateRecord**

The coordinate record is shown in Figure 7 and includes a printRecord method to output the coordinate classes various parameters. The _name parameter gives the name of the coordinate transformation and the remaining parameters are discussed in a previous part of this writeup.

```
class coordinateRecord:

    def __init__(self, _class):
        self._name = _class._name
        self._params = _class._params
        self._vec = _class._vec
        self._inv_params = _class._inv_params
        self._inv_vec = _class._inv_vec

    def printRecord(self):
        print('name = ', self._name, '\n')
        print('_params = ', self._params, '\n')
        print('_vec = ', self._vec, '\n')
        print('_inv_params = ', self._inv_params, '\n')
        print('_inv_vec = ', self._inv_vec, '\n')
```

Figure 7

Running Matrices for Different Coordinate Transforms

We now run the matrices $A, B, E, and\ W$ for different coordinate transforms.

Polar From Cartesians

Equations (15) and (18) give the forward and reverse transformations from cartesian to polar.

$$x = r\cos(\theta)$$
$$y = r\sin(\theta) \tag{15}$$

where

$r > 0$
$0 \leq \theta \leq 2\pi$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) \tag{18}$$

The transform matrices from cartesian to polar coordinates generated by the code are given below.

$$A(polar, cartesian) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \tag{40}$$

$$B(polar, cartesian) = \begin{bmatrix} \cos(\theta) & -\frac{\sin(\theta)}{r} \\ \sin(\theta) & \frac{\cos(\theta)}{r} \end{bmatrix} \tag{41}$$

$$E(polar, cartesian) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \tag{42}$$

$$W(polar, cartesian) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} \tag{43}$$

PolarSqrt From Polar

The transform equations from polar to polarSqrt are shown in equations (44) and (45).

$$r = \sqrt{\bar{r}}$$
$$\theta = \sqrt{\bar{\theta}} \tag{44}$$

where

$$\bar{r} = r^2$$
$$\bar{\theta} = \theta^2 \tag{45}$$

where

$r > 0$
$0 \leq \theta \leq 2\pi$

$\bar{r} > 0$
$0 \leq \bar{\theta} \leq 4\pi^2$

The transform matrices from polar to polarSqrt coordinates are given below.

$$A(polarSqrt, polar) = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{46}$$

$$B(polarSqrt, polar) = \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix} \tag{47}$$

As stated above, the transformation $(polarSqrt, polar)$ assumes that the polar coordinates were derived from cartesian coordinates and is why we are using the $(polar, cartesian)$ basis vectors. We will compute the basis vectors and one forms by hand and then compare to the program output.

$$E(polarSqrt, polar) = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos(\theta)}{2\sqrt{\bar{r}}} & \frac{\sin(\theta)}{2\sqrt{\bar{r}}} \\ -\frac{r\sin(\theta)}{2\sqrt{\bar{\theta}}} & \frac{r\cos(\theta)}{2\sqrt{\bar{\theta}}} \end{bmatrix} = \begin{bmatrix} \frac{\cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} & \frac{\sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} \\ -\frac{\sqrt{\bar{r}}\sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} & \frac{\sqrt{\bar{r}}\cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{48}$$

The output of the program is given by equation (49) ⇒

$$E(polarSqrt, polar) = \begin{bmatrix} \frac{\cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} & \frac{\sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} \\ -\frac{\sqrt{\bar{r}}\sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} & \frac{\sqrt{\bar{r}}\cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{49}$$

Equations (48) and (49) are the same.

$$W(polarSqrt, polar) = \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} = \begin{bmatrix} 2\sqrt{\bar{r}}\cos(\theta) & 2\sqrt{\bar{r}}\cos(\theta) \\ -\frac{2\sqrt{\bar{\theta}}\sin(\theta)}{r} & \frac{2\sqrt{\bar{\theta}}\cos(\theta)}{r} \end{bmatrix}$$

$$\begin{bmatrix} 2\sqrt{\bar{r}}\cos\left(\sqrt{\bar{\theta}}\right) & 2\sqrt{\bar{r}}\cos\left(\sqrt{\bar{\theta}}\right) \\ -\frac{2\sqrt{\bar{\theta}}\sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} & \frac{2\sqrt{\bar{\theta}}\cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \end{bmatrix}$$

(50)

Equation (51) is the output from the program ⇒

$$W(polarSqrt, polar) = \begin{bmatrix} 2\sqrt{\bar{r}}\cos\left(\sqrt{\bar{\theta}}\right) & 2\sqrt{\bar{r}}\sin\left(\sqrt{\bar{\theta}}\right) \\ -\frac{2\sqrt{\bar{\theta}}\sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} & \frac{2\sqrt{\bar{\theta}}\cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \end{bmatrix}$$

(51)

Equations (50) and (51) are the same.

Polar1 From Polar

The transformation equations are shown equations (52) and (53).

$$r = \bar{r} + \bar{\theta}$$
$$\theta = \bar{r} - \bar{\theta}$$

(52)

$$\bar{r} = \frac{r + \theta}{2}$$

$$\bar{\theta} = \frac{r - \theta}{2}$$

(53)

where

$r > 0$
$0 \leq \theta \leq 2\pi$
$\bar{r} > 0$
$\bar{\theta} \geq 0$

Note: $\bar{\theta}$ can be transformed to the range $(0 \quad 2\pi)$ by subtracting $2\pi n$ for an appropriate n.

$$A(polar1, polar) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(54)

$$B(polar1, polar) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \tag{55}$$

We will compute the basis vectors and one forms by hand and then compare to the program output.

$$E(polar1, polar) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) - r\sin(\theta) & \sin(\theta) + r\cos(\theta) \\ \cos(\theta) + r\sin(\theta) & \sin(\theta) - r\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\bar{r}-\bar{\theta}) - (\bar{r}+\bar{\theta})\sin(\bar{r}-\bar{\theta}) & \sin(\bar{r}-\bar{\theta}) + (\bar{r}+\bar{\theta})\cos(\bar{r}-\bar{\theta}) \\ \cos(\bar{r}-\bar{\theta}) + (\bar{r}+\bar{\theta})\sin(\bar{r}-\bar{\theta}) & \sin(\bar{r}-\bar{\theta}) - (\bar{r}+\bar{\theta})\cos(\bar{r}-\bar{\theta}) \end{bmatrix} \tag{56}$$

Equation (57) is the output from the program.

$$E(polar1, polar) = \begin{bmatrix} (\bar{\theta}+\bar{r})\sin(\bar{\theta}-\bar{r}) + \cos(\bar{\theta}-\bar{r}) & (\bar{\theta}+\bar{r})\cos(\bar{\theta}-\bar{r}) - \sin(\bar{\theta}-\bar{r}) \\ -(\bar{\theta}+\bar{r})\sin(\bar{\theta}-\bar{r}) + \cos(\bar{\theta}-\bar{r}) & -(\bar{\theta}+\bar{r})\cos(\bar{\theta}-\bar{r}) - \sin(\bar{\theta}-\bar{r}) \end{bmatrix} \tag{57}$$

Equation (56) is the same equation (57) but the sympy simplification changed $(\bar{r}-\bar{\theta})$ to $(\bar{\theta}-\bar{r})$ which is why there are some differences in the signs.

$$W(polar1, polar) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} = \begin{bmatrix} \frac{\cos(\theta)}{2} - \frac{\sin(\theta)}{2r} & \frac{\sin(\theta)}{2} + \frac{\cos(\theta)}{2r} \\ \frac{\cos(\theta)}{2} + \frac{\sin(\theta)}{2r} & \frac{\sin(\theta)}{2} - \frac{\cos(\theta)}{2r} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos(\bar{r}-\bar{\theta})}{2} - \frac{\sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} & \frac{\sin(\bar{r}-\bar{\theta})}{2} + \frac{\cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} \\ \frac{\cos(\bar{r}-\bar{\theta})}{2} + \frac{\sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} & \frac{\sin(\bar{r}-\bar{\theta})}{2} - \frac{\cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} \end{bmatrix} \tag{58}$$

Simplifying each element of equation (58) ⇒

$$W_{11} = \frac{\cos(\bar{r}-\bar{\theta})}{2} - \frac{\sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} = \frac{(\bar{r}+\bar{\theta})\cos(\bar{r}-\bar{\theta}) - \sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})}$$

$$W_{12} = \frac{\sin(\bar{r}-\bar{\theta})}{2} + \frac{\cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} = \frac{(\bar{r}+\bar{\theta})\sin(\bar{r}-\bar{\theta}) + \cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})}$$

$$W_{21} = \frac{\cos(\bar{r}-\bar{\theta})}{2} + \frac{\sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} = \frac{(\bar{r}+\bar{\theta})\cos(\bar{r}-\bar{\theta}) + \sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})}$$

$$W_{22} = \frac{\sin(\bar{r}-\bar{\theta})}{2} - \frac{\cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} = \frac{(\bar{r}+\bar{\theta})\sin(\bar{r}-\bar{\theta}) - \cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} \Rightarrow$$

$$W(polar1, polar) = \begin{bmatrix} \frac{(\bar{r}+\bar{\theta})\cos(\bar{r}-\bar{\theta}) - \sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} & \frac{(\bar{r}+\bar{\theta})\sin(\bar{r}-\bar{\theta}) + \cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} \\ \frac{(\bar{r}+\bar{\theta})\cos(\bar{r}-\bar{\theta}) + \sin(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} & \frac{(\bar{r}+\bar{\theta})\sin(\bar{r}-\bar{\theta}) - \cos(\bar{r}-\bar{\theta})}{2(\bar{r}+\bar{\theta})} \end{bmatrix} \tag{59}$$

Equation (06) is the output from the program.

$$W(polar1, cartesian) = \begin{bmatrix} \frac{(\bar{\theta}+\bar{r})\cos(\bar{\theta}-\bar{r}) + \sin(\bar{\theta}-\bar{r})}{2(\bar{\theta}+\bar{r})} & \frac{-(\bar{\theta}+\bar{r})\sin(\bar{\theta}-\bar{r}) + \cos(\bar{\theta}-\bar{r})}{2(\bar{\theta}+\bar{r})} \\ \frac{(\bar{\theta}+\bar{r})\cos(\bar{\theta}-\bar{r}) - \sin(\bar{\theta}-\bar{r})}{2(\bar{\theta}+\bar{r})} & \frac{(-\bar{\theta}-\bar{r})\sin(\bar{\theta}-\bar{r}) - \cos(\bar{\theta}-\bar{r})}{2(\bar{\theta}+\bar{r})} \end{bmatrix} \tag{60}$$

Equation (60) is the output from the program and is the same as equation (59) but the sympy simplification changed $(\bar{r}-\bar{\theta})$ to $(\bar{\theta}-\bar{r})$ which accounts for the differences in signs.

PolarSqrt1 From Polar

The transformation equations are shown in equations (61) and (62).

$$r = \sqrt{\bar{r}} + \sqrt{\bar{\theta}}$$
$$\theta = \sqrt{\bar{r}} - \sqrt{\bar{\theta}} \tag{61}$$

$$\bar{r} = \left[\frac{r+\theta}{2}\right]^2$$

$$\bar{\theta} = \left[\frac{r-\theta}{2}\right]^2 \tag{62}$$

where

$r > 0$
$0 \leq \theta \leq 2\pi$
$\bar{r} > 0$
$\bar{\theta} \geq 0$

Note: $\bar{\theta}$ can be transformed to the range $(0 \quad 2\pi)$ by subtracting $2\pi n$ for an appropriate n.

Here are the analytic computations of the matrices.

$$A = \begin{bmatrix} \frac{\partial r}{\partial \bar{r}} & \frac{\partial \theta}{\partial \bar{r}} \\ \frac{\partial r}{\partial \bar{\theta}} & \frac{\partial \theta}{\partial \bar{\theta}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & \frac{1}{2\sqrt{\bar{r}}} \\ \frac{1}{2\sqrt{\bar{\theta}}} & -\frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{63}$$

The output from the program is given in equation (64) and is the same as equation (63).

$$A(polarSqrt1, polar) = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & \frac{1}{2\sqrt{\bar{r}}} \\ \frac{1}{2\sqrt{\bar{\theta}}} & -\frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{64}$$

$$B = \begin{bmatrix} \frac{\partial \bar{r}}{\partial r} & \frac{\partial \bar{\theta}}{\partial r} \\ \frac{\partial \bar{r}}{\partial \theta} & \frac{\partial \bar{\theta}}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{r+\theta}{2} & \frac{r-\theta}{2} \\ \frac{r+\theta}{2} & -\frac{r-\theta}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{\bar{r}} & \sqrt{\bar{\theta}} \\ \sqrt{\bar{r}} & -\sqrt{\bar{\theta}} \end{bmatrix} \tag{65}$$

The output from the program is given by equation (66) and is the same as equation (65).

$$B(polarSqrt1, polar) = \begin{bmatrix} \sqrt{\bar{r}} & \sqrt{\bar{\theta}} \\ \sqrt{\bar{r}} & -\sqrt{\bar{\theta}} \end{bmatrix} \tag{66}$$

Equation (67) shows that $A \cdot B = I$

$$A \cdot B = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & \frac{1}{2\sqrt{\bar{r}}} \\ \frac{1}{2\sqrt{\bar{\theta}}} & -\frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \begin{bmatrix} \sqrt{\bar{r}} & \sqrt{\bar{\theta}} \\ \sqrt{\bar{r}} & -\sqrt{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{67}$$

Again, we use the polar basis vectors derived from cartesian coordinates.

$$E(polarSqrt1, polar) = \begin{bmatrix} \frac{1}{2\sqrt{\bar{\bar{r}}}} & \frac{1}{2\sqrt{\bar{\bar{r}}}} \\ \frac{1}{2\sqrt{\bar{\bar{\theta}}}} & -\frac{1}{2\sqrt{\bar{\bar{\theta}}}} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos(\theta) - r\sin(\theta)}{2\sqrt{\bar{\bar{r}}}} & \frac{\sin(\theta) + r\cos(\theta)}{2\sqrt{\bar{\bar{r}}}} \\ \frac{\cos(\theta) + r\sin(\theta)}{2\sqrt{\bar{\bar{\theta}}}} & \frac{\sin(\theta) - r\cos(\theta)}{2\sqrt{\bar{\bar{\theta}}}} \end{bmatrix}$$

Performing the following replacements: $r = \sqrt{\bar{\bar{r}}} + \sqrt{\bar{\bar{\theta}}}$ and $\theta = \sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}} \Rightarrow$

$$\begin{bmatrix} \frac{\cos\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right) - \left(\sqrt{\bar{\bar{r}}} + \sqrt{\bar{\bar{\theta}}}\right)\sin\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right)}{2\sqrt{\bar{\bar{r}}}} & \frac{\sin\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right) + \left(\sqrt{\bar{\bar{r}}} + \sqrt{\bar{\bar{\theta}}}\right)\cos\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right)}{2\sqrt{\bar{\bar{r}}}} \\ \frac{\cos\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right) + \left(\sqrt{\bar{\bar{r}}} + \sqrt{\bar{\bar{\theta}}}\right)\sin\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right)}{2\sqrt{\bar{\bar{\theta}}}} & \frac{\sin\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right) - \left(\sqrt{\bar{\bar{r}}} + \sqrt{\bar{\bar{\theta}}}\right)\cos\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right)}{2\sqrt{\bar{\bar{\theta}}}} \end{bmatrix} \tag{68}$$

Equation (69) is the output from the program and is the same as equation (68) but the sympy simplification changed $\left(\sqrt{\bar{\bar{r}}} - \sqrt{\bar{\bar{\theta}}}\right)$ to $\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right)$ which accounts for the sign differences.

$$E(polarSqrt1, polar) = \begin{bmatrix} \frac{\left(\sqrt{\bar{\bar{\theta}}} + \sqrt{\bar{\bar{r}}}\right)\sin\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right) + \cos\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right)}{2\sqrt{\bar{\bar{r}}}} & \frac{\left(\sqrt{\bar{\bar{\theta}}} + \sqrt{\bar{\bar{r}}}\right)\cos\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right) - \sin\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right)}{2\sqrt{\bar{\bar{r}}}} \\ \frac{-\left(\sqrt{\bar{\bar{\theta}}} + \sqrt{\bar{\bar{r}}}\right)\sin\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right) + \cos\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right)}{2\sqrt{\bar{\bar{\theta}}}} & \frac{\left(-\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right)\cos\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right) - \sin\left(\sqrt{\bar{\bar{\theta}}} - \sqrt{\bar{\bar{r}}}\right)}{2\sqrt{\bar{\bar{\theta}}}} \end{bmatrix} \tag{69}$$

$$W(polarSqrt1, polar) = \begin{bmatrix} \sqrt{\bar{\bar{r}}} & \sqrt{\bar{\bar{r}}} \\ \sqrt{\bar{\bar{\theta}}} & -\sqrt{\bar{\bar{\theta}}} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} =$$

$$= \begin{bmatrix} \sqrt{\bar{\bar{r}}}\cos(\theta) - \sqrt{\bar{\bar{r}}}\frac{\sin(\theta)}{r} & \sqrt{\bar{\bar{r}}}\sin(\theta) - \sqrt{\bar{\bar{r}}}\frac{\cos(\theta)}{r} \\ \sqrt{\bar{\bar{\theta}}}\cos(\theta) + \sqrt{\bar{\bar{\theta}}}\frac{\sin(\theta)}{r} & \sqrt{\bar{\bar{\theta}}}\sin(\theta) - \sqrt{\bar{\bar{\theta}}}\frac{\cos(\theta)}{r} \end{bmatrix}$$

$$
= \begin{bmatrix} \frac{r\sqrt{\bar{r}}\cos(\theta) - \sqrt{\bar{r}}\sin(\theta)}{r} & \frac{r\sqrt{\bar{r}}\sin(\theta) - \sqrt{\bar{r}}\cos(\theta)}{r} \\ \frac{r\sqrt{\bar{\theta}}\cos(\theta) + \sqrt{\bar{\theta}}\sin(\theta)}{r} & \frac{r\sqrt{\bar{\theta}}\sin(\theta) - \sqrt{\bar{\theta}}\cos(\theta)}{r} \end{bmatrix}
$$

Performing the following replacements: $r = \sqrt{\bar{r}} + \sqrt{\bar{\theta}}$ and $\theta = \sqrt{\bar{r}} - \sqrt{\bar{\theta}} \Rightarrow$

$$
\begin{bmatrix} \frac{\sqrt{\bar{r}}\left[\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)\cos\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right) - \sin\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right)\right]}{\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)} & \frac{\sqrt{\bar{r}}\left[\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)\sin\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right) - \cos\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right)\right]}{\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)} \\ \frac{\sqrt{\bar{\theta}}\left[\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)\cos\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right) + \sin\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right)\right]}{\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)} & \frac{\sqrt{\bar{\theta}}\left[\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)\sin\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right) - \cos\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right)\right]}{\left(\sqrt{\bar{r}} + \sqrt{\bar{\theta}}\right)} \end{bmatrix} \tag{70}
$$

Equation (71) is generated from the software.

$$
W(polarSqrt1, polar) = \begin{bmatrix} \frac{\sqrt{\bar{r}}\left(\left(\sqrt{\bar{\theta}} + \sqrt{\bar{r}}\right)\cos\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right) + \sin\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right)\right)}{\sqrt{\bar{\theta}} + \sqrt{\bar{r}}} & \frac{\sqrt{\bar{r}}\left(-\left(\sqrt{\bar{\theta}} + \sqrt{\bar{r}}\right)\sin\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right) + \cos\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right)\right)}{\sqrt{\bar{\theta}} + \sqrt{\bar{r}}} \\ \frac{\sqrt{\bar{\theta}}\left(\left(\sqrt{\bar{\theta}} + \sqrt{\bar{r}}\right)\cos\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right) - \sin\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right)\right)}{\sqrt{\bar{\theta}} + \sqrt{\bar{r}}} & \frac{\sqrt{\bar{\theta}}\left(\left(-\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right)\sin\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right) - \cos\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right)\right)}{\sqrt{\bar{\theta}} + \sqrt{\bar{r}}} \end{bmatrix} \tag{71}
$$

Equation (71) is the same as equation (70) but the sympy simplification changed $\left(\sqrt{\bar{r}} - \sqrt{\bar{\theta}}\right)$ to $\left(\sqrt{\bar{\theta}} - \sqrt{\bar{r}}\right)$ which accounts for the sign differences.