

Coordinates Summary

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Introduction

This writeup provides a coordinate summary for reference. It is updated from earlier write ups in several ways. First, it explicitly includes the basis vectors. This is helpful when we look at geometric invariants where the area or volume of the basis set needs to be considered. Second, while earlier writeups were correct, the notation for the linear algebra was mixed and used both row and column vectors as default depending on the situation. In this writeup, row vectors are the default.

Basis Vectors

Given a set of basis vectors which are linearly independent and span the space, we can write an arbitrary vector as a linear combination of the basis. “Linear independence” means that no two basis vectors are parallel. “Span the space” means that any vector in the space can be written in terms of the basis vectors. Note: the basis vectors are not assumed to be orthogonal or normalized. If there are n basis vectors, then an arbitrary vector will have n components.

A general basis set will be represented by a matrix as shown in equation (1).

$$E = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & \ddots & \vdots \\ e_{n1} & \cdots & e_{nn} \end{bmatrix} \quad (1)$$

where

each row of E is a basis vector
the default vector is a row vector

In a general basis, a vector can be represented by equation (2).

$$\mathbf{v} = \mathbf{v}^i E \quad (2)$$

where

$\mathbf{v}^i$ is a row vector of components in the basis E .

Equation (3) expands equation (2).

$$v^i E = v^1 \mathbf{e}_1 + \dots + v^n \mathbf{e}_n = v^1 [e_{11} \quad \dots \quad e_{1n}] + v^2 [e_{21} \quad \dots \quad e_{2n}] + \dots + v^n [e_{n1} \quad \dots \quad e_{nn}] \quad (3)$$

The basis vector components themselves need coordinates which are typically cartesian.

Example 1

Let the basis vectors in a two-dimensional space be the following:

$$\begin{aligned} \mathbf{e}_1 &= [\sqrt{3} \quad 1] \\ \mathbf{e}_2 &= [2 \quad 2\sqrt{3}] \end{aligned} \quad (4)$$

These basis vectors are not orthogonal or normalized. Note: they are given in cartesian coordinates,

$$E = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} \quad (5)$$

Pick a vector in this coordinate system $\Rightarrow$

$$u^i = [1 \quad 2]$$

Multiplying by basis vectors - E - gives $\mathbf{u}$ in cartesian coordinates.

$$\mathbf{u} = u^i E = [1 \quad 2] \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} = [\sqrt{3} + 4 \quad 4\sqrt{3} + 1] \quad (6)$$

Pick another vector in this coordinate system

$$v^i = [3 \quad 4]$$

$$\mathbf{v} = v^i E = [3 \quad 4] \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} = [3\sqrt{3} + 8 \quad 8\sqrt{3} + 3] \quad (7)$$

Equation (7) gives $\mathbf{v}$ in cartesian coordinates.

Dot Product

Compute the dot product in a general basis. Define two vectors as shown in equation (8).

$$\mathbf{u} = u^i E$$

$$\mathbf{v} = v^i E$$

(8)

Taking the dot product of these two vectors $\Rightarrow$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \mathbf{v}^T = u^i E (v^i E)^T = u^i E E^T (v^i)^T = u^i G (v^i)^T$$

(9)

Equation (9) gives the formula for a dot product in a general basis. The expression EE^T is called the metric and is denoted by G . It gives a measure of distance. This can be seen by using the dot product to find the magnitude of a vector.

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u^i G (u^i)^T}$$

(10)

Example 2

$$u^i = [1 \quad 2]$$

$$v^i = [3 \quad 4]$$

$$G = EE^T = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2 \\ 1 & 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 4 & 4\sqrt{3} \\ 4\sqrt{3} & 16 \end{bmatrix}$$

(11)

$$\mathbf{u} \cdot \mathbf{v} = u^i G (v^i)^T = [1 \quad 2] \begin{bmatrix} 4 & 4\sqrt{3} \\ 4\sqrt{3} & 16 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [1 \quad 2] \begin{bmatrix} 12 + 16\sqrt{3} \\ 12\sqrt{3} + 64 \end{bmatrix} =$$

$$= 12 + 16\sqrt{3} + 24\sqrt{3} + 128 = 40\sqrt{3} + 140$$

(12)

Computing the dot product in cartesian coordinates from equations (6) and (7) $\Rightarrow$

$$\mathbf{u} \cdot \mathbf{v} = [\sqrt{3} + 4 \quad 4\sqrt{3} + 1] \begin{bmatrix} 3\sqrt{3} + 8 \\ 8\sqrt{3} + 3 \end{bmatrix} = (\sqrt{3} + 4)(3\sqrt{3} + 8) + (4\sqrt{3} + 1)(8\sqrt{3} + 3)$$

$$= 9 + 8\sqrt{3} + 12\sqrt{3} + 32 + 96 + 12\sqrt{3} + 8\sqrt{3} + 3 = 40\sqrt{3} + 140$$

As expected, this is the same result as equation (12).

We can create a new basis W called a reciprocal basis to simplify the dot product calculation by removing the need for G in equation (9). Replacing E by W in equation (9) and lowering the index on u to indicate that its components are in the reciprocal basis $\Rightarrow$

$$\mathbf{u} \cdot \mathbf{v} = u_i W E^T (v^i)^T \quad (13)$$

Now set $W E^T = I \Rightarrow$

$$W = (E^T)^{-1} = (E^{-1})^T \quad (14)$$

W is the reciprocal basis and u_i are the components of $\mathbf{u}$ in that basis.

Equation (13) $\Rightarrow$

$$\mathbf{u} \cdot \mathbf{v} = u_i (v^i)^T \quad (15)$$

$$W E^T = (E^{-1})^T E^T = I = I^T = E E^{-1} = E W^T \quad (16)$$

Using equation (16), the dot product can be written as

$$\mathbf{u} \cdot \mathbf{v} = u^i E W^T (v_i)^T = u^i (v_i)^T \quad (17)$$

Equation (17) simply represents the dot product by raising u 's index and by lowering v 's index.

The dot product can also be computed using the reciprocal basis components only.

$$\begin{aligned} \mathbf{u} &= u_i W \\ \mathbf{v} &= v_i W \end{aligned} \quad (18)$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \mathbf{v}^T = u_i W (v_i W)^T = u_i W W^T (v_i)^T \quad (19)$$

The expression $W W^T$ is called the inverse metric and is denoted by G^{-1}

Note: $G G^{-1} = E E^T W W^T = E W^T = I$

Equation (19) $\Rightarrow$

$$\mathbf{u} \cdot \mathbf{v} = u_i G^{-1}(v_i)^T = u^i (v_i)^T \quad (20)$$

Again, length can be defined in terms of a vector magnitude.

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_i G^{-1}(u_i)^T} \quad (21)$$

Notation

Here is a summary of the notation.

1. E is a matrix where the rows are basis vectors
2. Basis vectors can also be notated as $\mathbf{e}_i$
3. Superscripted un-bolded variables are vector components - u^i
4. W is a reciprocal basis where $WE^T = I$
5. Reciprocal basis vectors can be written as $\boldsymbol{\omega}^i$
6. Subscripted un-bolded variables are reciprocal vector components - u_i
7. g_{ij} are the components of the metric
8. g^{ij} are the components of the inverse metric

For example, the vector $\mathbf{u}$ can be represented in either basis as

$$\mathbf{u} = u^i E = u^i \mathbf{e}_i = u_i W = \boldsymbol{\omega}^i u_i = u^i G W \quad (22)$$

In equation (22), we use the Einstein summation convention where there is an implied sum over the same index that occurs in the upper and lower positions. So, equation (22) would be equivalent to

$$\mathbf{u} = \sum_i u^i \mathbf{e}_i = \sum_i u_i \boldsymbol{\omega}^i \quad (23)$$

Index Raising and Lowering

From equations (9) and (15), the dot product can be written as

$$\mathbf{u} \cdot \mathbf{v} = u_i (v^i)^T = u^i G (v^i)^T = u^i g_{ij} v^j \quad (24)$$

From equation (24), we see that

$$u_i = u^i G$$

so G lowers an upper index.

From equations (19) the dot product can be written as

$$\mathbf{u} \cdot \mathbf{v} = u^i (v_i)^T = u_i G^{-1} (v_i)^T \quad (25)$$

$$u^i = u_i G^{-1} \quad (26)$$

So G^{-1} raises a lower index.

Example 3

Using E from the previous example $\Rightarrow$

$$W = (E^{-1})^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} \end{bmatrix}^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \quad (27)$$

$$u_i = u^i G = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 4\sqrt{3} \\ 4\sqrt{3} & 16 \end{bmatrix} = \begin{bmatrix} 4 + 8\sqrt{3} & 4\sqrt{3} + 32 \end{bmatrix} \quad (28)$$

Equation (22) $\Rightarrow$

$$\begin{aligned} u_i W &= \begin{bmatrix} 4 + 8\sqrt{3} & 4\sqrt{3} + 32 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{3}}{2} [4 + 8\sqrt{3}] - \frac{1}{4} [4\sqrt{3} + 32] & -\frac{1}{2} [4 + 8\sqrt{3}] + \frac{\sqrt{3}}{4} [4\sqrt{3} + 32] \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{3} + 4 & 4\sqrt{3} + 1 \end{bmatrix} \end{aligned} \quad (29)$$

Equation (29) is the same as equation (6) as expected.

Vector Under a Change in Coordinates

Suppose the basis changes from E to $\bar{E}$ as shown in equation (30).

$$\bar{E} = AE \tag{30}$$

where

$$E = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & \ddots & \vdots \\ e_{n1} & \cdots & e_{nn} \end{bmatrix}$$

$$\bar{E} = \begin{bmatrix} \bar{\mathbf{e}}_1 \\ \vdots \\ \bar{\mathbf{e}}_n \end{bmatrix} = \begin{bmatrix} \bar{e}_{11} & \cdots & \bar{e}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{e}_{n1} & \cdots & \bar{e}_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \equiv \text{coordinate transformation matrix.}$$

as before, the basis vectors are the rows of E and $\bar{E}$.

Equation (30) maps the basis vectors of E to those of $\bar{E}$.

The vector $\mathbf{v}$ is a geometric object that is independent of the basis $\Rightarrow$ it's represented in $\Rightarrow$

$$\mathbf{v} = v^i E = \bar{v}^i \bar{E} \tag{31}$$

where v^i and $\bar{v}^i$ are row vectors of components in the basis E and $\bar{E}$ respectively

The idea behind equation (31) is to have $\bar{v}^i$ and $\bar{E}$ change in a way to keep the geometric object $\mathbf{v}$ the same. Equation (30) relates the E basis to the $\bar{E}$ basis. The next step is to relate $\bar{v}^i$ and v^i .

$$\bar{v}^i = v^i B \tag{32}$$

where

B is a matrix to be determined

To determine B , put equations (30) and (32) into equation (31) $\Rightarrow$

$$\mathbf{v} = v^i E = \bar{v}^i \bar{E} = v^i B A E \Rightarrow \tag{33}$$

To make equation (33) an equality $\Rightarrow$

$$BA = I \Rightarrow$$

$$B = A^{-1} \quad (34)$$

To show how the reciprocal basis vectors transform, invert equation (30) $\Rightarrow$

$$\begin{aligned} \bar{E}^{-1} &= (AE)^{-1} = \bar{W}^T = E^{-1}A^{-1} = W^TB \Rightarrow \\ \bar{W} &= [W^TB]^T = B^TW \end{aligned} \quad (35)$$

To show a reciprocal vector under a change of coordinates $\Rightarrow$

$$\bar{v}_i = v_i C \quad (36)$$

where C is to be determined

$$\mathbf{v} = v_i W = \bar{v}_i \bar{W} = v_i C B^T W \quad (37)$$

To make equation (37) an equality $\Rightarrow$

$$C B^T = I = C [A^{-1}]^T = C [A^T]^{-1} \Rightarrow$$

$$C = A^T \Rightarrow$$

$$\bar{v}_i = v_i A^T \quad (38)$$

Dot Product Under a Change in Coordinates

The dot product is an invariant, so it should be the same regardless of what coordinate system it is in. From equation (30)

$$E = A^{-1} \bar{E} = B \bar{E} \quad (39)$$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u^i E (v^i E)^T = u^i G (v^i)^T = u^i (v_i)^T = u^i B \bar{E} (v^i B \bar{E})^T = \bar{u}^i \bar{E} (\bar{v}^i \bar{E})^T = \bar{u}^i \bar{G} (\bar{v}^i)^T \\ &= \bar{u}^i (\bar{v}_i)^T = \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} \end{aligned} \quad (40)$$

Equation (40) shows that the dot product is invariant under a coordinate transformation as was expected.

Metric Under a Change in Coordinates

The metric transforms as shown in equation (41)

$$\bar{G} = \bar{E}\bar{E}^T = (AE)(AE)^T = AEE^TA^T = AGA^T \quad (41)$$

The inverse metric transforms as shown in equation (42).

$$\bar{G}^{-1} = \bar{W}\bar{W}^T = B^TW(B^TW)^T = B^TWW^TB = B^TG^{-1}B \quad (42)$$

Curvilinear Coordinates

A derivative can be considered a tangent vector and the properties of derivatives can be used to define a coordinate transform using the derivatives as basis functions.

Coordinate definitions.

$$q^1(\bar{q}^1 \quad \dots \quad \bar{q}^n) \Rightarrow$$

$$\vdots$$

$$q^n(\bar{q}^1 \quad \dots \quad \bar{q}^n) \Rightarrow$$

Coordinate definitions in vector form

$$q^i(\bar{q}^j) \Rightarrow \mathbf{q}(\bar{\mathbf{q}}) \quad (43)$$

Definitions of coordinate transform from properties of derivatives $\Rightarrow$

$$\frac{\partial}{\partial \bar{q}^1} = \frac{\partial q^1}{\partial \bar{q}^1} \frac{\partial}{\partial q^1} + \dots + \frac{\partial q^n}{\partial \bar{q}^1} \frac{\partial}{\partial q^n}$$

$$\vdots$$

$$\frac{\partial}{\partial \bar{q}^n} = \frac{\partial q^1}{\partial \bar{q}^n} \frac{\partial}{\partial q^1} + \dots + \frac{\partial q^n}{\partial \bar{q}^n} \frac{\partial}{\partial q^n}$$

(44)

where

$$\bar{\mathbf{e}}_i = \frac{\partial}{\partial \bar{q}^i} \equiv \text{basis vectors in } \bar{q}^i \text{ coordinates}$$

$$\mathbf{e}_i = \frac{\partial}{\partial q^i} \equiv \text{basis vectors in } q^i \text{ coordinates}$$

Equation (44) in matrix form $\Rightarrow$

$$\frac{\partial}{\partial \bar{\mathbf{q}}} = \frac{\partial \mathbf{q}}{\partial \bar{\mathbf{q}}} \frac{\partial}{\partial \mathbf{q}} \Rightarrow \bar{E} = AE \quad (45)$$

This has the same form as equation (30)

where

$$\begin{aligned} \bar{E} &= \frac{\partial}{\partial \bar{\mathbf{q}}} = \begin{bmatrix} \bar{\mathbf{e}}_1 \\ \vdots \\ \bar{\mathbf{e}}_n \end{bmatrix} \\ E &= \frac{\partial}{\partial \mathbf{q}} = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix} \\ A &= \begin{bmatrix} \frac{\partial \mathbf{q}}{\partial \bar{\mathbf{q}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial q^1}{\partial \bar{q}^1} & \cdots & \frac{\partial q^n}{\partial \bar{q}^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial q^1}{\partial \bar{q}^n} & \cdots & \frac{\partial q^n}{\partial \bar{q}^n} \end{bmatrix} \end{aligned} \quad (46)$$

Equation (46) is the definition of a matrix derivative¹. Note – this definition, is given in numerator layout notation.

Inverse coordinate definitions in coordinate form $\Rightarrow$

$$\bar{q}^1(q^1 \quad \cdots \quad q^n)$$

$$\vdots$$

$$\bar{q}^n(q^1 \quad \cdots \quad q^n) \Rightarrow$$

Inverse coordinate definitions in matrix form $\Rightarrow$

¹ [Matrix Calculus](#)

$$\bar{q}^j(q^i) \Rightarrow \bar{\mathbf{q}}(\mathbf{q}) \quad (47)$$

$$\begin{aligned} d\bar{q}^1 &= \frac{\partial \bar{q}^1}{\partial q^1} dq^1 + \dots + \frac{\partial \bar{q}^1}{\partial q^n} dq^n \\ &\vdots \\ d\bar{q}^n &= \frac{\partial \bar{q}^n}{\partial q^1} dq^1 + \dots + \frac{\partial \bar{q}^n}{\partial q^n} dq^n \end{aligned} \quad (48)$$

Notice that equation (48) defines the transpose of a matrix derivative – see equation (46) and swap the $\bar{q}^i$ and q^i terms to see that the i and j indices need to be swapped. Putting equation (48) in matrix form $\Rightarrow$

$$\mathbf{d}\bar{\mathbf{q}} = \left[\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{q}} \right]^T \mathbf{d}\mathbf{q} \Rightarrow \bar{\mathbf{W}} = \mathbf{B}^T \mathbf{W} \quad (49)$$

Equation (49) has the same matrix form as equation (35).

where

$$\bar{\mathbf{W}} = \mathbf{d}\bar{\mathbf{q}}$$

$$\mathbf{B} = \frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{q}}$$

$$\mathbf{W} = \mathbf{d}\mathbf{q}$$

Now

$$\mathbf{W} = \mathbf{d}\mathbf{q} = \begin{bmatrix} dq^1 \\ \vdots \\ dq^n \end{bmatrix}$$

$$\mathbf{E} = \frac{\partial}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial}{\partial q^1} \\ \vdots \\ \frac{\partial}{\partial q^n} \end{bmatrix}$$

$$\bar{\mathbf{W}} = \mathbf{d}\bar{\mathbf{q}} = \begin{bmatrix} d\bar{q}^1 \\ \vdots \\ d\bar{q}^n \end{bmatrix}$$

$$\bar{E} = \frac{\partial}{\partial \bar{q}} = \begin{bmatrix} \frac{\partial}{\partial \bar{q}^1} \\ \vdots \\ \frac{\partial}{\partial \bar{q}^n} \end{bmatrix} \Rightarrow$$

$$W^T E = [dq^1 \quad \dots \quad dq^n] \begin{bmatrix} \frac{\partial}{\partial q^1} \\ \vdots \\ \frac{\partial}{\partial q^n} \end{bmatrix} = \frac{\partial q^1}{\partial q^1} + \dots + \frac{\partial q^n}{\partial q^n} = I = \frac{\partial \bar{q}^1}{\partial \bar{q}^1} + \dots + \frac{\partial \bar{q}^n}{\partial \bar{q}^n} = [d\bar{q}^1 \quad \dots \quad d\bar{q}^n] \begin{bmatrix} \frac{\partial}{\partial \bar{q}^1} \\ \vdots \\ \frac{\partial}{\partial \bar{q}^n} \end{bmatrix}$$

$$= \bar{W}^T \bar{E} = I$$

(50)

$$BA = \frac{\partial \bar{q}}{\partial q} \frac{\partial q}{\partial \bar{q}} = \frac{\partial \bar{q}}{\partial \bar{q}} \frac{\partial q}{\partial q} = I$$

(51)

Example 4

Let's transform from cartesian coordinates to the coordinate system of Example 1 – equation (5)
 $\Rightarrow$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$G = EE^T = II^T = I$$

$$\bar{E} = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} E = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} I$$

(52)

Equation (52) $\Rightarrow$

$$A = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix}$$

(53)

$$\bar{G} = AGA^T = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix}^T = \begin{bmatrix} \sqrt{3} & 1 \\ 2 & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2 \\ 1 & 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 4 & 4\sqrt{3} \\ 4\sqrt{3} & 16 \end{bmatrix}$$

(54)

As expected, we get the same result as equation (11).

From Example 3 $\Rightarrow$

$$\bar{W} = (\bar{E}^{-1})^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \quad (55)$$

Since $W = I \Rightarrow$

$$\bar{W} = B^T W \Rightarrow$$

$$B^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \quad (56)$$

$$G^{-1} = WW^T = II^T = I$$

$$\bar{G}^{-1} = B^T G^{-1} B \Rightarrow$$

$$\bar{G}^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} I \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \quad (57)$$

To check $\Rightarrow$

$$G = [\bar{G}^{-1}]^{-1} = \begin{bmatrix} 1 & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 4\sqrt{3} \\ 4\sqrt{3} & 16 \end{bmatrix} \quad (58)$$

As expected, this is the same as equation (54).

Summary Equations

Figure 1 shows the relationships between the various basis vectors and vectors. Notice that the basis vectors are transposed to keep the transforms consistent as shown below. The default vector is a row vector, and the basis vectors are rows of the basis matrices as previously discussed. All vectors and basis vectors operate on the left side of the transform.

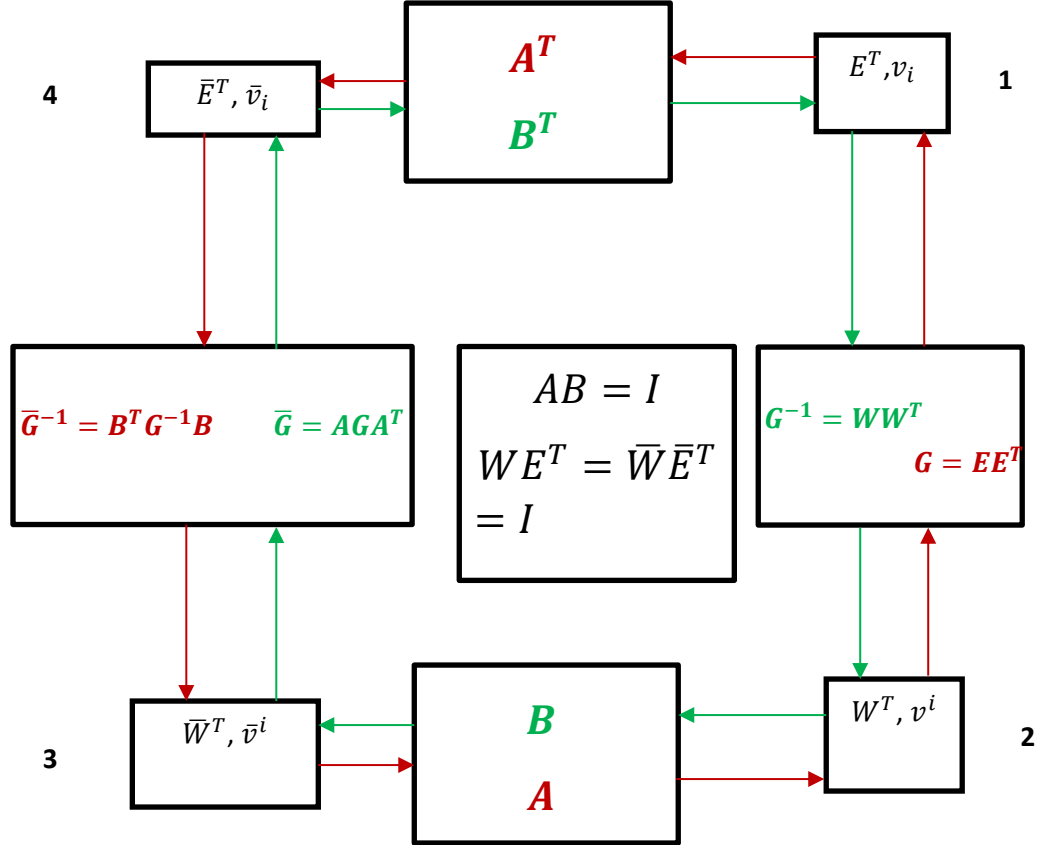


Figure 1

We will go through the paths between nodes of Figure 1.

Equations between nodes 1 and 4

$$\bar{v}_i = \bar{v}^i \bar{G} = v^i B \bar{G} = v^i B A G A^T = v^i G A^T = v_i A^T \quad (59)$$

$$\bar{E} = AE \Rightarrow$$

$$\bar{E}^T = E^T A^T \quad (60)$$

Equations (59) and (60) give a consistent transform of A^T between nodes 1 and 4.

Equations between nodes 4 and 1

$$v_i = \bar{v}_i [A^T]^{-1} = \bar{v}_i B^T \quad (61)$$

$$\bar{E} = AE \Rightarrow$$

$$E = A^{-1} \bar{E} = B \bar{E}$$

$$\text{because } B = A^{-1}$$

$$\text{Transposing } E \Rightarrow$$

$$E^T = \bar{E}^T B^T \quad (62)$$

Equations (61) and (62) give a consistent transform of B^T between nodes 4 and 1.

Equations between nodes 1 and 2

$$v^i = v_i G^{-1} \quad (63)$$

$$E^T G^{-1} = E^T W W^T = W^T \Rightarrow \quad (64)$$

Equations (63) and (64) give a consistent transform of G^{-1} between nodes 1 and 2.

Equations between nodes 2 and 1

$$v_i = v^i G \quad (65)$$

$$E = [W^T]^{-1} = [E^T G^{-1}]^{-1} = G [E^T]^{-1} = G W \Rightarrow$$

$$E^T = [G W]^T = W^T G^T = W^T G \quad (66)$$

$$\text{Because the metric is symmetric } \Rightarrow G = G^T$$

Equations (65) and (66) give a consistent transform of G between nodes 2 and 1.

Equations between nodes 4 and 3

$$\bar{v}^i = \bar{v}_i \bar{G}^{-1} = \bar{v}_i B^T G^{-1} B \quad (67)$$

$$\bar{W}^T = \bar{E}^T \bar{G}^{-1} = \bar{E}^T B^T G^{-1} B \quad (68)$$

Equations (67) and (68) give a consistent transform of $\bar{G}^{-1}$ between nodes 4 and 3.

Equations between nodes 3 and 4

$$\bar{v}_i = \bar{v}^i \bar{G} = \bar{v}^i A G A^T \quad (69)$$

$$\bar{W}^T = \bar{E}^T \bar{G}^{-1} \Rightarrow$$

$$\bar{E}^T = \bar{W}^T \bar{G} = \bar{W}^T A G A^T \quad (70)$$

Equations (69) and (70) give a consistent transform of $\bar{G}$ between nodes 3 and 4.

Equations between nodes 2 and 3

$$\bar{v}^i = v^i B \quad (71)$$

$$\bar{W} = B^T W$$

$$\bar{W}^T = [B^T W]^T = W^T B \quad (72)$$

Equations (71) and (72) give a consistent transform of B between nodes 2 and 3.

Equations between nodes 3 and 2

$$v^i = \bar{v}^i B^{-1} = \bar{v}^i A \quad (73)$$

$$W^T = \bar{W}^T B^{-1} = \bar{W}^T A \quad (74)$$

Equations (73) and (74) give a consistent transform of A between nodes 3 and 2.