Covariant Derivative in Matrix Form
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10/26/2021

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Introduction

This write-up develops the covariant derivative in matrix form which builds from the Christoffel Symbols in Matrix Form writeup.

Covariant Derivative in Matrix Form

The matrix equation for a vector is given by equation (1).

$$\boldsymbol{v} = v^i \boldsymbol{e_i} = v^1[e_{11} \quad \cdots \quad e_{1n}] + \cdots + v^n[e_{n1} \quad \cdots \quad e_{nn}] = \begin{bmatrix} v^1 e_{11} + \cdots + v^n e_{n1} \\ \vdots \\ v^1 e_{1n} + \cdots + v^n e_{nn} \end{bmatrix}$$
$$= [v^1 \quad \cdots \quad v^n]E = v^i E \tag{1}$$

where

v^i is a row vector of components - v^i
E is matrix of matrix of basis vectors where each row is a basis vector

Taking the derivative of (1) with respect to coordinate q^p. ⇒

$$\frac{\partial \boldsymbol{v}}{\partial q^p} = \frac{\partial \left(v^i E\right)}{\partial q^p} = \frac{\partial v^i}{\partial q^p} E + v^i \frac{\partial E}{\partial q^p} \tag{3}$$

But

$$\frac{\partial E}{\partial q^p} = \Gamma_p E \tag{4}$$

See [1]

[1] Christoffel Symbols in Matrix Form

Plugging equation (4) into equation (3) ⇒

$$\frac{\partial \boldsymbol{v}}{\partial q^p} = \frac{\partial v^i}{\partial q^p} E + v^i \Gamma_p E \Rightarrow$$

$$\frac{\partial \boldsymbol{v}}{\partial q^p} = \left[\frac{\partial v^i}{\partial q^p} + v^i \Gamma_p\right] E \tag{5}$$

Factoring out the basis vectors - E - as was done in Curvilinear Calculus[2] ⇒

$$\nabla_p \boldsymbol{v} = \frac{\partial v^i}{\partial q^p} + v^i \Gamma_p \tag{6}$$

Equation (6) is the covariant derivative in matrix form.

where

v^i is a row vector of components

$$\Gamma_p = \begin{bmatrix} {\mathit{\Gamma}_{1p}}^1 & \cdots & {\mathit{\Gamma}_{1p}}^n \\ \vdots & \ddots & \vdots \\ {\mathit{\Gamma}_{np}}^1 & \cdots & {\mathit{\Gamma}_{np}}^n \end{bmatrix} = \left[\mathit{\Gamma}_{ip}^k\right]$$

$\nabla_p \boldsymbol{v}$ is a row vector

Matrix Parameterized Covariant Derivative

The parameterized covariant derivative is given by equation (7).

The coordinates are parameterized as shown below.

$q^p(\lambda) \Rightarrow$

$$\frac{\partial \boldsymbol{v}}{\partial \lambda} = \frac{\partial \boldsymbol{v}}{\partial q^p}\frac{\partial q^p}{\partial \lambda} = \frac{\partial q^p}{\partial \lambda}\nabla_p \boldsymbol{v} = \frac{\partial q^1}{\partial \lambda}\nabla_1 \boldsymbol{v} + \frac{\partial q^2}{\partial \lambda}\nabla_2 \boldsymbol{v} + \cdots + \frac{\partial q^n}{\partial \lambda}\nabla_n \boldsymbol{v} \tag{7}$$

[2] Curvilinear Calculus

where

$\nabla_p \boldsymbol{v}$ is given by equation (6) and is a row vector

q^p is a row vector of coordinates

Equation (8) shows the matrix form of the parameterized covariant derivative shown in equation (7).

$$\frac{\partial \boldsymbol{v}}{\partial \lambda} = \nabla_i \boldsymbol{v} \left[\frac{\partial q^i}{\partial \lambda}\right]^T \tag{8}$$

where

$\nabla_i \boldsymbol{v}$ is a row vector of covariant derivatives

$\frac{\partial q^i}{\partial \lambda}$ is a row vector of the derivative of coordinates with respect to the parameter λ

Equation (8) is a dot product.

Matrix Covariant Derivative for the 2-D Sphere

The block matrix for the Christoffel Symbols of the Second Kind for the 2-D sphere is shown in equation (9).

$$\Gamma = \Gamma_{ij}^{k} = \begin{bmatrix}\Gamma_{i\theta}^{k} & \Gamma_{i\varphi}^{k}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}\Gamma_{\theta\theta}^{\theta} & \Gamma_{\theta\theta}^{\varphi} \\ \Gamma_{\varphi\theta}^{\theta} & \Gamma_{\varphi\theta}^{\varphi}\end{bmatrix} & \begin{bmatrix}\Gamma_{\theta\varphi}^{\theta} & \Gamma_{\theta\varphi}^{\varphi} \\ \Gamma_{\varphi\varphi}^{\theta} & \Gamma_{\varphi\varphi}^{\varphi}\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}\begin{bmatrix}0 & 0 \\ 0 & \frac{cos(\theta)}{sin(\theta)}\end{bmatrix} & \begin{bmatrix}0 & \frac{cos(\theta)}{sin(\theta)} \\ -cos(\theta)sin(\theta) & 0\end{bmatrix}\end{bmatrix} \tag{9}$$

Equations (10) and (11) show the covariance derivatives for a vector on the surface of a 2-D sphere.

$$\nabla_\theta \boldsymbol{v} = \frac{\partial v^k}{\partial \theta} + \Gamma_\theta v^k = \frac{\partial v^k}{\partial \theta} + v^k \begin{bmatrix}0 & 0 \\ 0 & \frac{cos(\theta)}{sin(\theta)}\end{bmatrix} \tag{10}$$

$$\nabla_\varphi \boldsymbol{v} = \frac{\partial v^k}{\partial \varphi} + \Gamma_\varphi v^k = \frac{\partial v^k}{\partial \varphi} + v^k \begin{bmatrix}0 & \frac{cos(\theta)}{sin(\theta)} \\ -cos(\theta)sin(\theta) & 0\end{bmatrix} \tag{11}$$

Check Parameterized Matrix Covariant Derivative for the 2-D Sphere

We can check the parameterized matrix covariant derivatives by using the resulting vectors from the Parallel Transport write up[3] and showing that the covariant derivative of these vectors is zero. Note that the parallel transport write-up has not been updated to the default vector being a row vector. The results are still valid.

Constant Longitude Case

Equation (12) shows the equation for a parallel transported vector along a constant longitude on the 2-D sphere.

$$\boldsymbol{v}(\theta) = \begin{bmatrix} v_0^\theta & \dfrac{v_0^\varphi \sin(\theta_0)}{\sin(\theta)} \end{bmatrix} \tag{12}$$

where

$\varphi = \varphi_0 \equiv$constant

θ changes along a constant longitude path $\Rightarrow \lambda = \theta$

$$\frac{\partial[q^p]}{\partial\lambda} = \frac{\partial}{\partial\theta}\begin{bmatrix}\theta \\ \varphi\end{bmatrix} = \begin{bmatrix}1 \\ 0\end{bmatrix} \Rightarrow$$

$$\frac{\partial\boldsymbol{v}}{\partial\lambda} = [\nabla_\theta\boldsymbol{v} \quad \nabla_\varphi\boldsymbol{v}]\begin{bmatrix}1 \\ 0\end{bmatrix} = \nabla_\theta\boldsymbol{v} \tag{13}$$

$$\frac{\partial\boldsymbol{v}}{\partial\lambda} = \nabla_\theta\boldsymbol{v} = \frac{\partial[v^k]}{\partial\theta} + v^k\begin{bmatrix} 0 & 0 \\ 0 & \dfrac{cos(\theta)}{sin(\theta)} \end{bmatrix} \tag{14}$$

Plugging equation (12) into equation (14) $\Rightarrow$

$$\begin{bmatrix} 0 & -\dfrac{v_0^\varphi \sin(\theta_0) cos(\theta)}{\sin^2(\theta)} \end{bmatrix} + \begin{bmatrix} 0 & cot(\theta)\dfrac{v_0^\varphi \sin(\theta_0)}{\sin(\theta)} \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{v_0^\varphi \sin(\theta_0) cos(\theta)}{\sin^2(\theta)} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & \dfrac{v_0^\varphi \sin(\theta_0) cos(\theta)}{\sin^2(\theta)} \end{bmatrix} = [0 \quad 0] \tag{15}$$

These results are what were expected.

[3] Parallel Transport

Constant Latitude Case

Equation (16) shows the equations for a parallel transported vector along a constant latitude.

$\theta_0 \equiv$ constant
φ changes along a constant latitude path $\Rightarrow \lambda = \varphi$

$$v^\theta(\theta_0, \varphi) = v^\theta(\theta_0, 0)\cos(k\varphi) + \sin(\theta_0)v^\varphi(\theta_0, 0)\sin(k\varphi)$$

$$v^\varphi(\theta_0, \varphi) = v^\varphi(\theta_0, 0)\cos(k\varphi) - \frac{v^\theta(\theta_0, 0)}{\sin(\theta_0)}\sin(k\varphi)$$

(16)

where $k = \cos(\theta_0)$

$$[v^k] = [v^\theta(\theta_0, \varphi) \quad v^\varphi(\theta_0, \varphi)]$$

$$\frac{\partial [q^p]}{\partial \lambda} = \frac{\partial}{\partial \varphi}\begin{bmatrix}\theta \\ \varphi\end{bmatrix} = \begin{bmatrix}0 \\ 1\end{bmatrix} \Rightarrow$$

$$\frac{\partial \boldsymbol{v}}{\partial \lambda} = [\nabla_\theta \boldsymbol{v} \quad \nabla_\varphi \boldsymbol{v}]\begin{bmatrix}0 \\ 1\end{bmatrix} = \nabla_\varphi \boldsymbol{v} \Rightarrow$$

$$\nabla_\varphi \boldsymbol{v} = \frac{\partial v^k}{\partial \varphi} + v^k[\Gamma_\varphi] = \frac{\partial v^k}{\partial \varphi} + v^k\begin{bmatrix}0 & \dfrac{cos(\theta_0)}{sin(\theta_0)} \\ -cos(\theta_0)sin(\theta_0) & 0\end{bmatrix}$$

$$= \frac{\partial v^k}{\partial \varphi} + \left[-cos(\theta_0)sin(\theta_0)v^\varphi \quad \frac{cos(\theta_0)}{sin(\theta_0)}v^\theta\right]$$

(17)

The components of equation (17) get long in this case, so we break the calculations up into their θ and φ components.

Theta Component

$$\frac{\partial v^{\theta}(\theta_0,\varphi)}{\partial \varphi} = -v^{\theta}(\theta_0,0)\sin(k\varphi)cos(\theta_0) + \sin(\theta_0)v^{\varphi}(\theta_0,0)\cos(k\varphi)cos(\theta_0) \tag{18}$$

$$\{v^k\Gamma_{\varphi}\}_{\theta} = -cos(\theta_0)sin(\theta_0)v^{\varphi} = -cos(\theta_0)sin(\theta_0)\left(v^{\varphi}(\theta_0,0)\cos(k\varphi) - \frac{v^{\theta}(\theta_0,0)}{\sin(\theta_0)}\sin(k\varphi)\right)$$
$$= -cos(\theta_0)sin(\theta_0)v^{\varphi}(\theta_0,0)\cos(k\varphi) + cos(\theta_0)sin(\theta_0)\frac{v^{\theta}(\theta_0,0)}{\sin(\theta_0)}\sin(k\varphi) \tag{19}$$

Adding equations (18) and (19) ⇒

$$\frac{\partial v^{\theta}(\theta_0,\varphi)}{\partial \varphi} + \{v^k\Gamma_{\varphi}\}_{\theta} = -v^{\theta}(\theta_0,0)\sin(k\varphi)cos(\theta_0) + \sin(\theta_0)v^{\varphi}(\theta_0,0)\cos(k\varphi)cos(\theta_0)$$

$$-cos(\theta_0)sin(\theta_0)v^{\varphi}(\theta_0,0)\cos(k\varphi) + cos(\theta_0)v^{\theta}(\theta_0,0)\sin(k\varphi) = 0 \tag{20}$$

which is what is expected.

Phi Component

$$\frac{\partial v^{\varphi}(\theta_0,\varphi)}{\partial \varphi} = -v^{\varphi}(\theta_0,0)\sin(k\varphi)cos(\theta_0) - \frac{v^{\theta}(\theta_0,0)}{\sin(\theta_0)}cos(k\varphi)cos(\theta_0) \tag{21}$$

$$\{[v^k][\Gamma_{\varphi}]\}_{\varphi} = \frac{cos(\theta_0)}{sin(\theta_0)}v^{\theta} = \frac{cos(\theta_0)}{sin(\theta_0)}\left[v^{\theta}(\theta_0,0)\cos(k\varphi) + \sin(\theta_0)v^{\varphi}(\theta_0,0)\sin(k\varphi)\right]$$

$$= \frac{cos(\theta_0)}{sin(\theta_0)}v^{\theta}(\theta_0,0)\cos(k\varphi) + v^{\varphi}(\theta_0,0)\sin(k\varphi)cos(\theta_0) \tag{22}$$

Adding equations (21) and (22) ⇒

$$\frac{\partial v^{\varphi}(\theta_0,\varphi)}{\partial \varphi} + \{v^k\Gamma_{\varphi}\}_{\varphi}$$
$$= -v^{\varphi}(\theta_0,0)\sin(k\varphi)cos(\theta_0) - \frac{v^{\theta}(\theta_0,0)}{\sin(\theta_0)}cos(k\varphi)cos(\theta_0)$$
$$+ \frac{cos(\theta_0)}{sin(\theta_0)}v^{\theta}(\theta_0,0)\cos(k\varphi) + v^{\varphi}(\theta_0,0)\sin(k\varphi)cos(\theta_0) = 0$$

(23)

which is what is expected

Putting the θ and φ elements together $\Rightarrow$

$$\nabla_{\varphi}\boldsymbol{v} = [0 \quad 0] \tag{24}$$