Geometric Invariants

By
Al Bernstein
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http://www.metricmath.com
al@metricmath.com

Introduction

This write-up discusses geometric invariants with regards to tensors. Figure 1 shows the relationships between the various basis vectors and vector components.[1] Notice that the basis vectors are transposed to keep the transforms consistent as shown below. The default vector is a row vector, and the basis vectors are rows of the basis matrices. A vector or matrix is applied on the left of an operator.

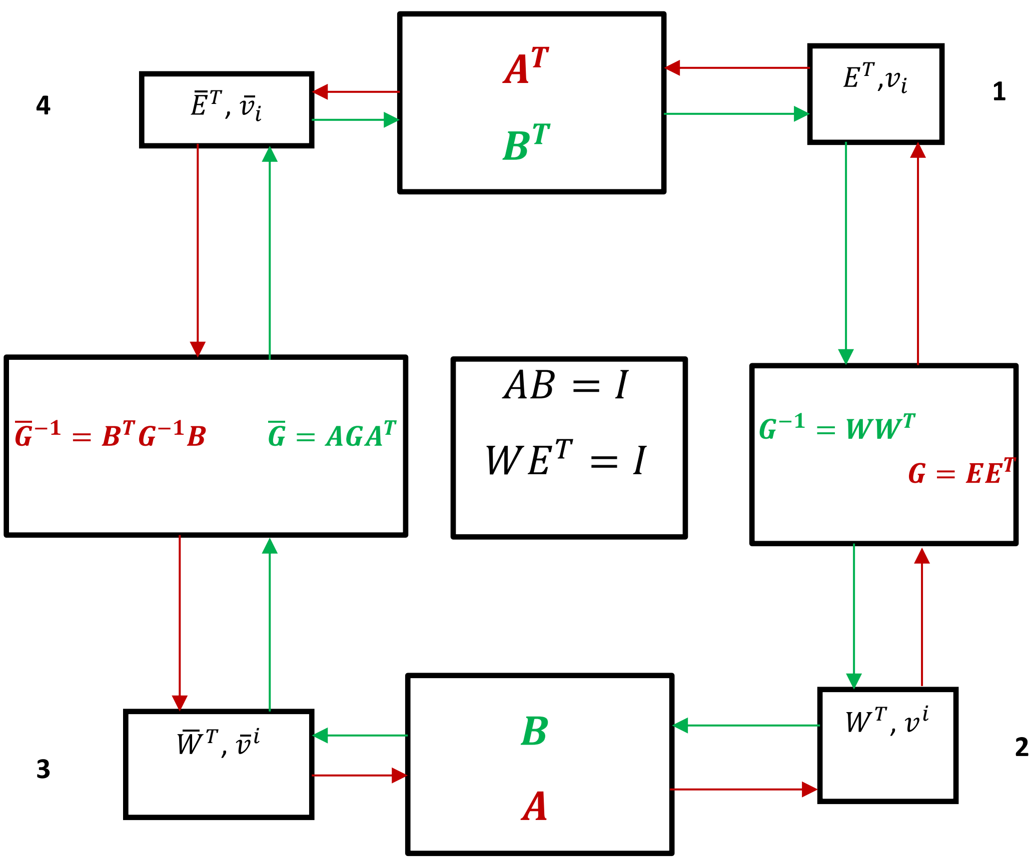

Figure 1

[1] Coordinates Summary

Equation (1) shows the definitions of a vector in terms of its basis vectors.

$$\boldsymbol{v} = v^i \boldsymbol{e_i} = v^1[e_{11} \quad \cdots \quad e_{1n}] + \cdots + v^n[e_{n1} \quad \cdots \quad e_{nn}] = \begin{bmatrix} v^1 e_{11} + \cdots + v^n e_{n1} \\ \vdots \\ v^1 e_{1n} + \cdots + v^n e_{nn} \end{bmatrix}$$
$$= [v^1 \quad \cdots \quad v^n]E = v^i E \tag{1}$$

where

v^i is a row vector of components
E is a matrix of basis vectors where each row is a basis vector
$\boldsymbol{e_i} = [e_{i1} \quad \cdots \quad e_{in}]$ is the i^{th} basis vector

Similarly, equation (2) gives the matrix equation for a vector in terms of a set of reciprocal basis vectors.

$$\boldsymbol{v} = v_i \boldsymbol{\omega^i} = v_i W \tag{2}$$

In Figure1, a vector or matrix operates on the left of a matrix transform as illustrated in the examples below.

Equations between nodes 1 and 4

$$\bar{v}_i = \bar{v}^i \bar{G} = v^i B \bar{G} = v^i BAGA^T = v^i GA^T = v_i A^T \tag{3}$$

$$\bar{E} = AE \Rightarrow$$

$$\bar{E}^T = E^T A^T \tag{4}$$

Equations between nodes 4 and 1

$$v_i = \bar{v}_i [A^T]^{-1} = \bar{v}_i B^T \tag{5}$$

$$\bar{E} = AE \Rightarrow$$

$$E = A^{-1}\bar{E} = B\bar{E}$$

because $B = A^{-1}$

Transposing $E \Rightarrow$

$$E^T = \bar{E}^T B^T \tag{6}$$

In general, a basis set does not have to be orthogonal or normalized. It does have to span the vector space meaning that any vector in the space can be written in terms of the basis set. When computing geometric invariants, properties of the basis set need to be taken into consideration.

Geometric Invariants

Inner Product

The matrices A and B were originally set up so that the inner product is invariant under a coordinate transform. Consider the following definitions of two vectors.

$\boldsymbol{u} = u_i W$ covariant vector components

$\boldsymbol{v} = v^i E$ contravariant vector components

It is necessary for one vector to have covariant components and one to have contravariant components.[2]

From Figure 1

$\bar{u}_i = u_i A^T$

$\bar{v}^i = v^i B$

Putting the above equations together and taking the inner product ⇒

$$\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{v}} = u_i A^T \bar{W} \left[v^i B \bar{E}\right]^T = u_i A^T \bar{W} \bar{E}^T \left[v^i B\right]^T = u_i A^T (\bar{E}^{-1})^T \bar{E}^T \left[v^i B\right]^T = u_i A^T \left[v^i B\right]^T$$

because $\bar{W} = (\bar{E}^{-1})^T$ – see [2]

[2] Coordinates Summary

Further simplifying ⇒

$$\overline{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}} = u_i A^T \left[v^i B\right]^T = u_i A^T B^T u_i \left(v^i\right)^T = u_i \left(v^i\right)^T = \boldsymbol{u} \cdot \boldsymbol{v}$$

because $B = A^{-1}$ see [2]

(7)

Equation (7) shows that the inner product is invariant under a coordinate transform as was expected. An application of an inner product is to compute the magnitude squared of a vector.

$$|\boldsymbol{v}|^2 = v^i \left(v^i\right)^T = v^i G \left[v^i\right]^T$$

(8)

where

$G = EE^T$ is the metric

$v^i \equiv$ contravariant vector components

Area

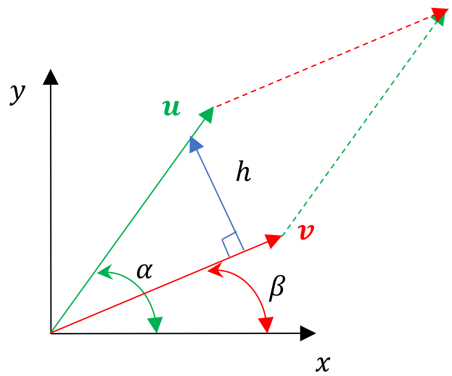

Figure 2

Figure 2 shows a parallelogram defined by 2 vectors in cartesian coordinates.

$$area = base \times height = |\boldsymbol{u}||\boldsymbol{v}| sin(\alpha - \beta) = |\boldsymbol{u}||\boldsymbol{v}|[sin(\alpha)cos(\beta) - sin(\beta)cos(\alpha)]$$

(9)

$$|\boldsymbol{u}| sin(\alpha) = u_y$$
$$|\boldsymbol{v}| cos(\beta) = v_x$$
$$|\boldsymbol{v}| sin(\beta) = v_y$$
$$|\boldsymbol{u}| cos(\alpha) = u_x$$

(10)

Equations (9) and (10) ⇒

$$signed\ area = u_y v_x - v_y u_x = u_x v_y - v_x u_y \tag{11}$$

Equation (11) can be looked at as a determinant. Define

$$M = VE = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} E = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$signed\ area = det(M) = det(VE) = det(V)det(E) = u_x v_y - v_x u_y \tag{12}$$

Equation (12) uses the properties of determinants[3] and gives a signed area in Cartesian coordinates. For general coordinates, transform from a Cartesian to a general coordinate system as shown in equation (13).

$$\bar{\boldsymbol{v}} = \bar{v}^i \bar{E}$$

$$\bar{\boldsymbol{u}} = \bar{u}^i \bar{E} \tag{13}$$

Create a matrix with vectors $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$ ⇒

$$\bar{M} = \bar{V}\bar{E} = \begin{bmatrix} \bar{u}_x & \bar{u}_x \\ \bar{v}_x & \bar{v}_y \end{bmatrix} \bar{E} = VBAE = VE \tag{14}$$

Equation (14) shows that the area is invariant under a transform as expected.

To see how the determinant transforms, take the determinant of equation (14) ⇒

$$det(\bar{M}) = det(\bar{V}\bar{E}) = det(\bar{V})det(\bar{E}) = det(V)det(E) \tag{15}$$

Area Example

For polar coordinates

$$x = r\ cos(\theta)$$
$$y = r\ sin(\theta)$$

$$A(r,\theta) = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -r\ sin(\theta) & r\ cos(\theta) \end{bmatrix}$$

[3] Determinants

$$(16)$$

Now make $A(r,\theta)$ an unnormalized transform⇒ use

$$A\left(10,\frac{\pi}{4}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -5\sqrt{2} & 5\sqrt{2} \end{bmatrix}$$

$$B\left(10,\frac{\pi}{4}\right) = A^{-1}\left(10,\frac{\pi}{4}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{5\cdot 2^{3/2}} & \frac{1}{5\cdot 2^{3/2}} \end{bmatrix}$$

Define two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in Cartesian coordinates ⇒

$$\boldsymbol{u} = \begin{bmatrix} 3 & 4 \end{bmatrix}$$
$$\boldsymbol{v} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$V = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$area = det(V) = 6 - 4 = 2 \tag{17}$$

Transform to the unnormalized polar coordinates using $B\left(10,\frac{\pi}{4}\right)$ ⇒

$$\bar{V} = \begin{bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{v}} \end{bmatrix} = VB = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{5\cdot 2^{3/2}} & \frac{1}{5\cdot 2^{3/2}} \end{bmatrix} = \begin{bmatrix} 2^{3/2} & -\frac{1}{5\sqrt{2}} \\ 3\sqrt{2} & -\frac{1}{5\sqrt{2}} \end{bmatrix}$$

$$\bar{E} = \begin{bmatrix} \bar{\boldsymbol{e}}_{\boldsymbol{1}} \\ \bar{\boldsymbol{e}}_{\boldsymbol{2}} \end{bmatrix} = AE = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{5\cdot 2^{3/2}} & \frac{1}{5\cdot 2^{3/2}} \end{bmatrix}$$

$$signed\ area = det(\bar{V})det(\bar{E}) = \frac{1}{5} \times 10 = 2$$

which is the same as equation (17) as expected

Area Calculations Using the Wedge Product

Given vectors

$\boldsymbol{a} = a^i \boldsymbol{e_i} = a^1 \boldsymbol{e_1} + a^2 \boldsymbol{e_2}$

$\boldsymbol{b} = b^i \boldsymbol{e_i} = b^1 \boldsymbol{e_1} + b^2 \boldsymbol{e_2}$

The wedge product for vectors is given by equation (18).

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a^1 \boldsymbol{e_1} + a^2 \boldsymbol{e_2}) \wedge (b^1 \boldsymbol{e_1} + b^2 \boldsymbol{e_2}) =$$

$$a^1 \boldsymbol{e_1} \wedge b^1 \boldsymbol{e_1} + a^1 \boldsymbol{e_1} \wedge b^2 \boldsymbol{e_2} + a^2 \boldsymbol{e_2} \wedge b^1 \boldsymbol{e_1} + a^2 \boldsymbol{e_2} \wedge b^2 \boldsymbol{e_2} \tag{18}$$

where

$\wedge$ is the wedge product operator

Note: the wedge product is distributive

To simplify equation (18),
use the following properties of the wedge product operator.

$$\boldsymbol{u} \wedge \boldsymbol{u} = \boldsymbol{0} \tag{19}$$

$$\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u} \tag{20}$$

Note determinants have the same properties as equations (19) and (20). Two identical rows or columns in a matrix will give a determinant of 0. Interchanging two rows or columns, reverses the determinant's sign.

Using equations (19) and (20) equation (18) $\Rightarrow$

$$\boldsymbol{a} \wedge \boldsymbol{b} = a^1 \boldsymbol{e_1} \wedge b^2 \boldsymbol{e_2} - b^1 \boldsymbol{e_1} \wedge a^2 \boldsymbol{e_2} = (a^1 b^2 - b^1 a^2) \boldsymbol{e_1} \wedge \boldsymbol{e_2} = det\left(\begin{bmatrix} a^1 & a^2 \\ b^1 & b^2 \end{bmatrix}\right) \boldsymbol{e_1} \wedge \boldsymbol{e_2} \tag{21}$$

$\boldsymbol{e_1} \wedge \boldsymbol{e_2}$ are called bivectors (2-vectors) and are a basis of coordinate plane segments spanned by $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$- e.g. $\{\hat{x}\hat{y}, \hat{x}\hat{z}, \hat{y}\hat{z}\}$[4]. A bivectors' magnitude is the area of the parallelogram defined by $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$.as shown in Figures 2a and 2b. The sign is determined by putting the tail of the second vector at the tip of the first vector. If the arrows are counterclockwise, the sign is positive, otherwise it's negative. Figure 2a shows a positive direction and Figure 2b shows a negative direction.

[4] Linear Algebra via Exterior Products - Sergei Winitzki, page 73

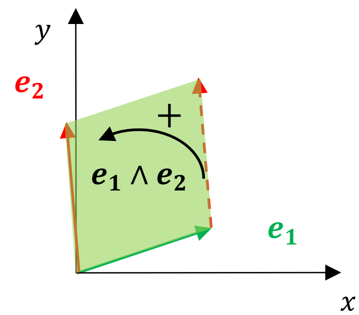

Figure 2a

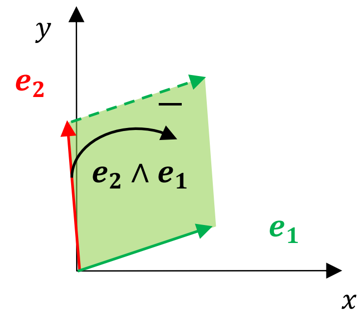

Figure 2b

The area calculation is given by equation (22).

$$
\begin{aligned}
\boldsymbol{\bar{u}} \wedge \boldsymbol{\bar{v}} &= (\bar{u}^1 \bar{\boldsymbol{e}}_1 + \bar{u}^2 \bar{\boldsymbol{e}}_2) \wedge (\bar{v}^1 \bar{\boldsymbol{e}}_1 + \bar{v}^2 \bar{\boldsymbol{e}}_2) \\
&= \bar{u}^1 \bar{v}^1 (\bar{\boldsymbol{e}}_1 \wedge \bar{\boldsymbol{e}}_1) + \bar{u}^1 \bar{v}^2 (\bar{\boldsymbol{e}}_1 \wedge \bar{\boldsymbol{e}}_2) + \bar{u}^2 \bar{v}^1 (\bar{\boldsymbol{e}}_2 \wedge \bar{\boldsymbol{e}}_1) + \bar{u}^2 \bar{v}^2 (\bar{\boldsymbol{e}}_2 \wedge \bar{\boldsymbol{e}}_2) \\
&= \bar{u}^1 \bar{v}^2 (\bar{\boldsymbol{e}}_1 \wedge \bar{\boldsymbol{e}}_2) + \bar{u}^2 \bar{v}^1 (\bar{\boldsymbol{e}}_2 \wedge \bar{\boldsymbol{e}}_1) = (\bar{u}^1 \bar{v}^2 - \bar{u}^2 \bar{v}^1) \bar{\boldsymbol{e}}_1 \wedge \bar{\boldsymbol{e}}_2
\end{aligned} \tag{22}
$$

Equation (22) is simply the determinant of $\bar{V}$ multiplied by $\bar{\boldsymbol{e}}_1 \wedge \bar{\boldsymbol{e}}_2$. The bivector is the determinant of $\bar{E}$ multiplied by $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$

$$
\bar{\boldsymbol{e}}_1 \wedge \bar{\boldsymbol{e}}_2 = det\left(\begin{bmatrix} \bar{\boldsymbol{e}}_1 \\ \bar{\boldsymbol{e}}_2 \end{bmatrix}\right) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \tag{23}
$$

$$
\boldsymbol{\bar{u}} \wedge \boldsymbol{\bar{v}} = det\left(\begin{bmatrix} u^i \\ v^i \end{bmatrix}\right) det\left(\begin{bmatrix} \bar{\boldsymbol{e}}_1 \\ \bar{\boldsymbol{e}}_2 \end{bmatrix}\right) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \tag{24}
$$

The determinant automatically takes care of the sign. Note that $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$ is considered a basis. Bivectors form a basis of coordinate plane segments using basis vectors, so equation (24) is the result.[5]

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[5] Linear Algebra via Exterior Products - Sergei Winitzki, page 77

Volume

The signed volume in three dimensions can be computed using three vectors that span the space as shown in equation (25).

Define three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in basis set $E \Rightarrow$

$$signed\ volume = \boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w} = det(V)det(E)\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \tag{25}$$

where

$$V = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \\ \boldsymbol{e_3} \end{bmatrix}$$

The basis set is now a tri-vector (3-vector) and are parallelepipeds.

Equation (25) can be extended to n dimensions.[6] The cross product is only valid in 3 and 7 dimensions.[7] The basis set of n-vectors is contained within $\mathbb{R}^n$.

$$signed\ volume\ in\ \mathbb{R}^n = \boldsymbol{v_1} \wedge \boldsymbol{v_2} \wedge \cdots \wedge \boldsymbol{v_n} = det(V)det(E)\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \cdots \boldsymbol{e_n}$$

$$V = \begin{bmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{v_n} \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \\ \vdots \\ \boldsymbol{e_n} \end{bmatrix}$$

$\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \cdots \boldsymbol{e_n}$ is an n-dimensional parallelepipeds basis

Note that in $\mathbb{R}^n$, an arbitrary object can be made from basis elements of $\mathbb{R}^n$ and subsets of $\mathbb{R}^n$.

Further Considerations of Wedge Products

An arbitrary wedge product can be written as shown in equation (26) and is called a k-vector

$$\boldsymbol{\alpha} = \boldsymbol{v_1} \wedge \boldsymbol{v_2} \wedge \cdots \wedge \boldsymbol{v_k} \tag{26}$$

The alternate property of equation (20) can be generalized as shown in equation (27).

Given a k-vector and a p-vector

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[6] Exterior Algebra

[7] Cross product

$$\boldsymbol{\alpha} = \boldsymbol{v_1} \wedge \boldsymbol{v_2} \wedge \cdots \wedge \boldsymbol{v_k}$$

$$\boldsymbol{\beta} = \boldsymbol{v_1} \wedge \boldsymbol{v_2} \wedge \cdots \wedge \boldsymbol{v_p}$$

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (-1)^{kp} \boldsymbol{\beta} \wedge \boldsymbol{\alpha} \tag{27}$$

The wedge product is associative ⇒

$$(\boldsymbol{u} \wedge \boldsymbol{v}) \wedge \boldsymbol{w} = \boldsymbol{u} \wedge (\boldsymbol{v} \wedge \boldsymbol{w}) \tag{28}$$

To understand equation (27) better, consider the wedge product of a bivector with a vector ⇒

$$(\boldsymbol{u} \wedge \boldsymbol{v}) \wedge \boldsymbol{w} = (-1)^{2\cdot 1} \boldsymbol{w} \wedge (\boldsymbol{u} \wedge \boldsymbol{v}) = \boldsymbol{w} \wedge (\boldsymbol{u} \wedge \boldsymbol{v}) \tag{29}$$

Equation (29) can also be derived by using equation (20) twice and using the associative property ⇒

$$(\boldsymbol{u} \wedge \boldsymbol{v}) \wedge \boldsymbol{w} = -(\boldsymbol{u} \wedge \boldsymbol{w}) \wedge \boldsymbol{v} = (\boldsymbol{w} \wedge \boldsymbol{u}) \wedge \boldsymbol{v} = \boldsymbol{w} \wedge (\boldsymbol{u} \wedge \boldsymbol{v}) \tag{30}$$