

Coordinate Transformation of Gradients and Partial Derivatives

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4/8/2022

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Introduction

This writeup investigates how gradients of scalar fields and partial derivatives of vector fields in 2-D behave under a change in coordinates. The scalar field is a function of coordinates and gives a number at each coordinate point. Because each point of the scalar field refers to a physical point, the scalar field is independent of any given coordinate system. An example of a scalar field is given in equation (1) that depends on the x coordinate only.

$$\varphi(x, y) = \varphi(\bar{x}, \bar{y}) = x^2 \tag{1}$$

where

$\varphi(x, y)$ and $\varphi(\bar{x}, \bar{y})$ are the scalar functions in coordinate systems (x, y) and $(\bar{x}, \bar{y})$ respectively

Transformation of basis vectors under a change of coordinates has been discussed in previous writeups.¹ Before continuing, it will be helpful to discuss coordinate transform equations and how they relate to coordinate grids and basis vectors.

Coordinate Transforms, Grids, and Basis Vectors

We use coordinate rotation as an example for this section.

Figure 1 shows a rectangular grid.

¹ [CoordinatesSummary](#)

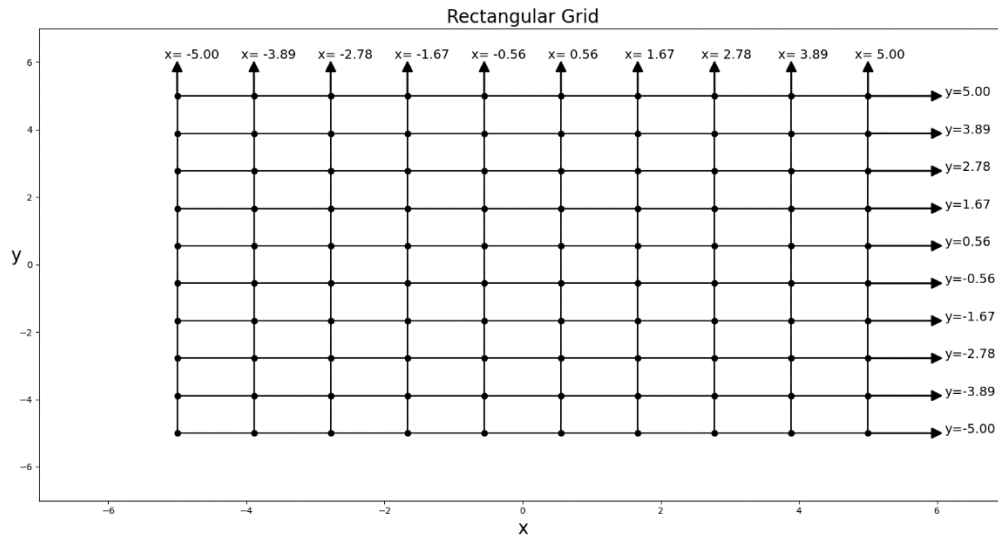


Figure 1

The grid lines are generated in the x direction by holding y constant and varying the x values. Holding x constant and changing y values will generate grid lines in the y direction. This idea may seem basic, but it is important to understand the procedure when the grids are not made up of straight lines. Figure 2 shows the grid from Figure 1 but rotated by $\frac{\pi}{3}$.

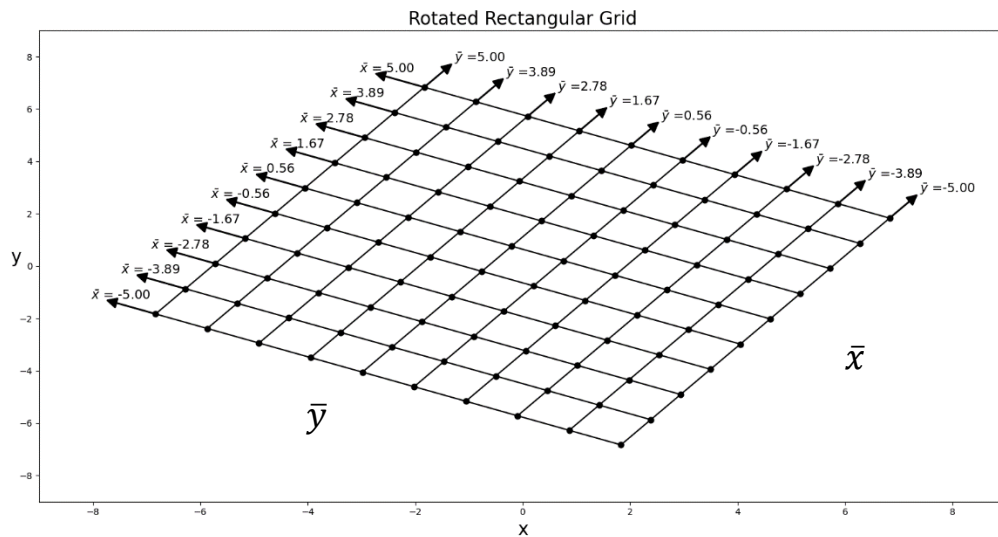


Figure 2

It's important to note that Figure 2 shows the rotated grid $(\bar{x}, \bar{y})$ graphed in (x, y) coordinates. A closeup of the rotated axes, $(\bar{x}, \bar{y})$ relative to the x and y axis is shown in Figures 3a and 3b.

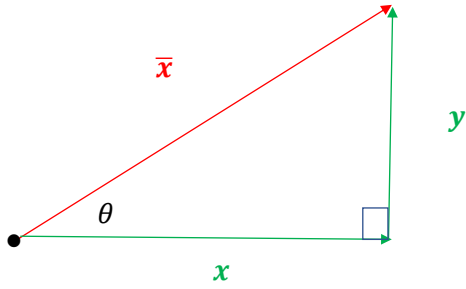


Figure 3a

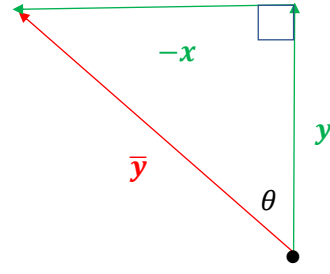


Figure 3b

In Figures 3a and 3b, the $\bar{x}$ and $\bar{y}$ axes are the hypotenuses of right triangles. Equation (2) gives the x and y coordinates as a function of the $\bar{x}$ and $\bar{y}$ coordinates.

$$\begin{aligned} x &= \bar{x}\cos(\theta) - \bar{y}\sin(\theta) \\ y &= \bar{x}\sin(\theta) + \bar{y}\cos(\theta) \end{aligned} \tag{2}$$

It's important to interpret equation (2) correctly. The input is the coordinate $(\bar{x}, \bar{y})$ - distance (numbers) along the $\bar{x}$ and $\bar{y}$ vectors respectively. The output is the coordinate (x, y) - distance (numbers) along the x and y vectors respectively.

Example

Given the following input - $(\bar{x}, \bar{y})$ coordinates and $\theta \Rightarrow$

$$(\bar{x}, \bar{y}) = (1, 0) \text{ and } \theta = \frac{\pi}{3}$$

The resulting (x, y) coordinates are given by equations (2)

$$\begin{aligned} x &= (1)\cos\left(\frac{\pi}{3}\right) + 0 = \frac{1}{2} \\ y &= (1)\sin\left(\frac{\pi}{3}\right) + 0 = \frac{\sqrt{3}}{2} \end{aligned} \tag{3a}$$

Equation (3a) shows that one unit along the $\bar{x}$ axis gives (x, y) coordinates of $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$.

For $(\bar{x}, \bar{y}) = (0, 1)$ and $\theta = \frac{\pi}{3}$

$$x = 0 - (1)\sin\left(\frac{\pi}{3}\right) + 0 = -\frac{\sqrt{3}}{2}$$

$$y = 0 + (1)\cos\left(\frac{\pi}{3}\right) + 0 = \frac{1}{2} \quad (3b)$$

Equation (3b) shows that one unit along the $\bar{y}$ axis gives (x, y) coordinates of $-\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$

Basis vectors transform as shown in equation (4)

$$\begin{aligned} \frac{\partial}{\partial \bar{x}} &= \frac{\partial x}{\partial \bar{x}} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{x}} \frac{\partial}{\partial y} = \cos(\theta) \frac{\partial}{\partial x} + \sin(\theta) \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \bar{y}} &= \frac{\partial x}{\partial \bar{y}} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{y}} \frac{\partial}{\partial y} = -\sin(\theta) \frac{\partial}{\partial x} + \cos(\theta) \frac{\partial}{\partial y} \end{aligned} \quad (4)$$

Equation (5) gives the A matrix defined by equation (4).

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \quad (5)$$

Equation (6) gives the B matrix.

$$B = A^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (6)$$

Figure 4a and 4b show the inverse coordinate relations for the rotation.

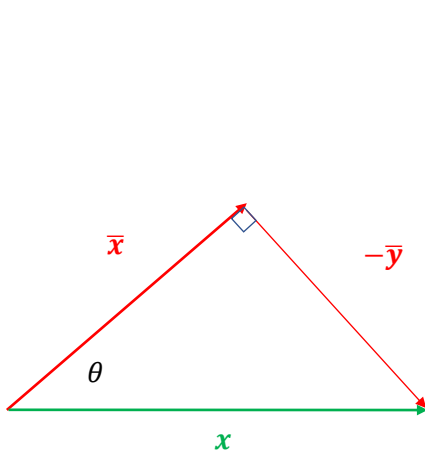


Figure 4a

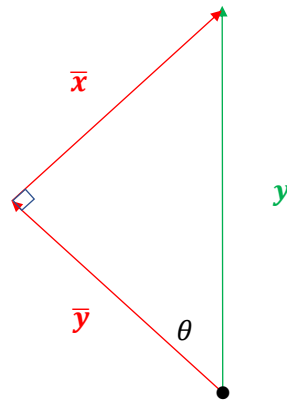


Figure 4b

The x and y axes are the hypotenuses of the right triangles in Figures 4a and 4b. Equation (7) gives the $(\bar{x}, \bar{y})$ coordinates as a function of the (x, y) coordinates.

$$\begin{aligned}\bar{x} &= x\cos(\theta) + y\sin(\theta) \\ \bar{y} &= -x\sin(\theta) + y\cos(\theta)\end{aligned}\tag{7}$$

Compute $\varphi(x, y)$ in the Rotated Frame

We want to describe the same scalar field in the original and rotated coordinate systems to show that $\varphi(\bar{x}, \bar{y}) = \varphi(x, y)$. The physical field consists of the same points but expressed in different coordinate systems. Using the definition from equation (1) and the relations from equation (2) $\Rightarrow$

$$\begin{aligned}\varphi(x, y) &= \varphi(\bar{x}, \bar{y}) = x^2 \Rightarrow \\ \varphi(\bar{x}, \bar{y}) &= (\bar{x}\cos(\theta) - \bar{y}\sin(\theta))^2\end{aligned}\tag{8}$$

For example, set

$$\theta = \frac{\pi}{3}$$

Equation (8) $\Rightarrow$

$$\varphi(\bar{x}, \bar{y}) = \left(\bar{x}\cos\left(\frac{\pi}{3}\right) - \bar{y}\sin\left(\frac{\pi}{3}\right)\right)^2 = \left(\frac{\bar{x}}{2} - \frac{\sqrt{3}}{2}\bar{y}\right)^2 = \frac{3\bar{y}^2}{4} - \frac{\sqrt{3}\bar{x}\bar{y}}{2} + \frac{\bar{x}^2}{4}\tag{9}$$

Example Calculation

$$x = 3, y = 0 \Rightarrow$$

$$\varphi(3, 0) = 9$$

$$\theta = \frac{\pi}{3}$$

$$\bar{x} = x\cos(\theta) + y\sin(\theta) = 3 \cdot \cos\left(\frac{\pi}{3}\right) = \frac{3}{2}$$

$$\bar{y} = -x\sin(\theta) + y\cos(\theta) = -3 \cdot \sin\left(\frac{\pi}{3}\right) = -\frac{3^{3/2}}{2}$$

(10)

Equation (10) show the $(\bar{x}, \bar{y})$ coordinates corresponding to $(x, y) = (3, 0)$

Inputting $\bar{x}$ and $\bar{y}$ from (10) into (9) $\Rightarrow$

$$\begin{aligned}\varphi\left(\frac{3}{2}, -\frac{3^{3/2}}{2}\right) &= \frac{3}{4}\left(-\frac{3^{3/2}}{2}\right)^2 - \frac{\sqrt{3}}{2}\left(\frac{3}{2}\right)\left(-\frac{3^{3/2}}{2}\right) + \frac{1}{4}\left(\frac{3}{2}\right)^2 \\ &= \frac{3}{4} \frac{27}{4} - \left(\frac{3^{3/2}}{4}\right)\left(-\frac{3^{3/2}}{2}\right) + \frac{1}{4}\left(\frac{3}{2}\right)^2 = \frac{81}{16} + \frac{27}{8} + \frac{9}{16} = \frac{81}{16} + \frac{54}{16} + \frac{9}{16} = \frac{144}{16} = 9\end{aligned}\tag{11}$$

which is what is expected.

Now create a table with a set of x points $(x, 0)$ and show the results for $\varphi(x, y)$ and $\varphi(\bar{x}, \bar{y})$.

x	y	$\bar{x}$	$\bar{y}$	$\varphi(x, y)$	$\varphi(\bar{x}, \bar{y})$
0	0	0	0	0	0
1	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	1	1
2	0	1	$-\sqrt{3}$	4	4
3	0	$\frac{3}{2}$	$-\frac{3^{3/2}}{2}$	9	9
4	0	2	$-2\sqrt{3}$	16	16
5	0	$\frac{5}{2}$	$-\frac{5}{2}\sqrt{3}$	25	25
6	0	3	$-3^{3/2}$	36	36
7	0	$\frac{7}{2}$	$-\frac{7}{2}\sqrt{3}$	49	49
8	0	4	$-4\sqrt{3}$	64	64
9	0	$\frac{9}{2}$	$-\frac{9}{2}\sqrt{3}$	81	81

Table 1

Test for negative $x \Rightarrow \varphi(-9, 0) = 81 \Rightarrow$

$$\begin{aligned}\bar{x} &= -9 \cdot \cos\left(\frac{\pi}{3}\right) = -\frac{9}{2} \\ \bar{y} &= -9 \cdot \sin\left(\frac{\pi}{3}\right) = \frac{9}{2}\sqrt{3}\end{aligned}$$

$$\begin{aligned}\bar{\varphi}\left(-\frac{9}{2}, \frac{9}{2}3^{3/2}\right) &= \frac{3}{4}\left(\frac{9}{2}\sqrt{3}\right)^2 - \frac{\sqrt{3}}{2}\left(-\frac{9}{2}\right)\left(\frac{9}{2}\sqrt{3}\right) + \frac{1}{4}\left(-\frac{9}{2}\right)^2 = \frac{729}{16} + \frac{486}{16} + \frac{81}{16} = \frac{1296}{16} \\ &= 81\end{aligned}$$

(12)

Compute Gradient in General Coordinates

Coordinate Relations for general coordinates are given by equation (13).

$$\begin{aligned} q^1(\bar{q}^1 \quad \dots \quad \bar{q}^n) &\Rightarrow \\ &\vdots \\ q^n(\bar{q}^1 \quad \dots \quad \bar{q}^n) \end{aligned} \quad (13)$$

Given a function $\varphi(q^1, q^2, \dots, q^n)$, the components of the gradient are given by

$$(\nabla\varphi)_i = \frac{\partial\varphi}{\partial q^i} \quad (14)$$

Looking at how the $\nabla\varphi$ components transform under a coordinate change $\Rightarrow$

$$\frac{\partial\varphi}{\partial \bar{q}^i} = \frac{\partial\varphi}{\partial q^j} \frac{\partial q^j}{\partial \bar{q}^i} \quad (15)$$

For $i = (1, 2)$, $j = (1, 2)$, and $q^i = (x, y) \Rightarrow$

$$(\nabla\varphi)_i = \begin{bmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{bmatrix}$$

Using the Einstein summation rule, equation (15) $\Rightarrow$

$$\begin{aligned} \frac{\partial\varphi}{\partial \bar{q}^1} &= \frac{\partial\varphi}{\partial q^1} \frac{\partial q^1}{\partial \bar{q}^1} + \frac{\partial\varphi}{\partial q^2} \frac{\partial q^2}{\partial \bar{q}^1} \Rightarrow \frac{\partial\varphi}{\partial \bar{x}} = \frac{\partial\varphi}{\partial x} \frac{\partial x}{\partial \bar{x}} + \frac{\partial\varphi}{\partial y} \frac{\partial y}{\partial \bar{x}} \\ \frac{\partial\varphi}{\partial \bar{q}^2} &= \frac{\partial\varphi}{\partial q^1} \frac{\partial q^1}{\partial \bar{q}^2} + \frac{\partial\varphi}{\partial q^2} \frac{\partial q^2}{\partial \bar{q}^2} \Rightarrow \frac{\partial\varphi}{\partial \bar{y}} = \frac{\partial\varphi}{\partial x} \frac{\partial x}{\partial \bar{y}} + \frac{\partial\varphi}{\partial y} \frac{\partial y}{\partial \bar{y}} \end{aligned} \quad (16)$$

Equation (16) $\Rightarrow$

$$\begin{bmatrix} \frac{\partial\varphi}{\partial \bar{x}} & \frac{\partial\varphi}{\partial \bar{y}} \end{bmatrix} = (\bar{\nabla}\varphi)_i = \begin{bmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \bar{x}} & \frac{\partial x}{\partial \bar{y}} \\ \frac{\partial y}{\partial \bar{x}} & \frac{\partial y}{\partial \bar{y}} \end{bmatrix} = \begin{bmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{bmatrix} A^T = (\nabla\varphi)_i A^T \quad (17)$$

where

$\bar{\nabla}$ is the gradient with respect to the $(\bar{x}, \bar{y})$ variables

Equation (17) is the prototype of a transformation of the components of a covariant vector from (x, y) to $(\bar{x}, \bar{y})$ coordinates as shown in equation (17)

$$\bar{v}_i = v_i A^T \quad (17)$$

So, if the components of $\nabla\varphi$ transform as a covariant vector, the bases must be one forms $\Rightarrow$

$$\nabla\varphi = \frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy \quad (18)$$

The gradient $(\nabla\varphi)_i$ can be solved for in terms of $(\bar{\nabla}\varphi)_i$ as shown in equation (19).

$$(\nabla\varphi)_i = (\bar{\nabla}\varphi)_i B^T \quad (19)$$

Example

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

For

$$\varphi(x, y) = x^2 \Rightarrow$$

$$\nabla\varphi(x, y) = [2x \quad 0]$$

$$\begin{aligned} (\bar{\nabla}\varphi)_i &= (\nabla\varphi)_i A^T = [2x \quad 0] \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}^T = [2x \quad 0] \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= [2x\cos(\theta) \quad -2x\sin(\theta)] \end{aligned} \quad (20)$$

We can convert equation (20) to $(\bar{x}, \bar{y})$ coordinates by using equation (2)

$$x = \bar{x}\cos(\theta) - \bar{y}\sin(\theta) \Rightarrow$$

$$\bar{\nabla}\varphi = [2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\cos(\theta) \quad -2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\sin(\theta)] \quad (21)$$

To check equation (21), compute $\bar{\nabla}\varphi$ straight from equation (2)

$$\varphi(\bar{x}, \bar{y}) = (\bar{x}\cos(\theta) - \bar{y}\sin(\theta))^2 \Rightarrow$$

$$\bar{\nabla}\varphi = [2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\cos(\theta) \quad -2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\sin(\theta)] \quad (22)$$

Equation (22) agrees with equation (21).

Using equation (19), we can transform $\bar{\nabla}\varphi$ back to $\nabla\varphi$.

$$B = A^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{aligned} \nabla\varphi &= \bar{\nabla}\varphi B^T \\ &= [2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\cos(\theta) \quad -2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\sin(\theta)] \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \Rightarrow \end{aligned}$$

Computing $(\nabla\varphi)_x$ and $(\nabla\varphi)_y$ separately $\Rightarrow$

$$\begin{aligned} (\nabla\varphi)_x &= [2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\cos^2(\theta) + 2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\sin^2(\theta)] \\ &= 2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)] \end{aligned}$$

$$(\nabla\varphi)_y = 2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\cos(\theta)\sin(\theta) - 2[\bar{x}\cos(\theta) - \bar{y}\sin(\theta)]\sin(\theta)\cos(\theta) = 0$$

Using equation (2) $\Rightarrow$

$$(\nabla\varphi)_x = 2x \Rightarrow$$

$$\nabla\varphi = [2x \quad 0] \quad (23)$$

as is expected.

Compute Gradient in Polar Coordinates

The coordinate relations are

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \quad (24)$$

$$A = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \quad (25)$$

The general equation for a change of basis for one-forms is given by

$$\mathbf{u} = \bar{u}_i \bar{W} = u_i W \quad (26)$$

The conversion equations are shown below

$$\bar{W} = B^T W \quad (27)$$

$$\bar{u}_i = u_i A^T \quad (28)$$

Using equation (28) $\Rightarrow$

$$\begin{aligned} (\bar{\nabla}\varphi)_i &= (\nabla\varphi)_i A^T = \begin{bmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial\varphi}{\partial x}\cos(\theta) + \frac{\partial\varphi}{\partial y}\sin(\theta) & -\frac{\partial\varphi}{\partial x}r\sin(\theta) + \frac{\partial\varphi}{\partial y}r\cos(\theta) \end{bmatrix} \end{aligned} \quad (29)$$

The coordinates in equation (29) can be converted to (r, θ) by using the chain rule as shown in equations (30).

$$\begin{aligned} \frac{\partial\varphi}{\partial r} &= \frac{\partial\varphi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial\varphi}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial\varphi}{\partial x} \cos(\theta) + \frac{\partial\varphi}{\partial y} \sin(\theta) \\ \frac{\partial\varphi}{\partial \theta} &= \frac{\partial\varphi}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial\varphi}{\partial y} \frac{\partial y}{\partial \theta} = -\frac{\partial\varphi}{\partial x} r\sin(\theta) + \frac{\partial\varphi}{\partial y} r\cos(\theta) \end{aligned} \quad (30)$$

Note: the A matrix is applying the chain rule, so it's automatically converting from (x, y) to (r, θ) coordinates – in general $(\bar{x}, \bar{y})$ coordinates.

Using equation (27) $\Rightarrow$

$$\begin{bmatrix} dr \\ d\theta \end{bmatrix} = B^T \begin{bmatrix} dx \\ dy \end{bmatrix} \quad (31)$$

Note: all the differentials are row vectors

$$B = A^{-1} = \begin{bmatrix} \cos(\theta) & -\frac{\sin(\theta)}{r} \\ \sin(\theta) & \frac{\cos(\theta)}{r} \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} dr \\ d\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \Rightarrow$$

$$\begin{aligned} dr &= \cos(\theta)dx + \sin(\theta)dy \\ d\theta &= -\frac{\sin(\theta)}{r}dx + \frac{\cos(\theta)}{r}dy \end{aligned} \quad (32)$$

Now write the basis vectors in equation (32) in terms of normalized basis vectors. Note: $dx = d\hat{x}$, and $dy = d\hat{y}$

$$\begin{aligned} dr &= |dr|d\hat{r} = |\cos(\theta)d\hat{x} + \sin(\theta)d\hat{y}|d\hat{r} \Rightarrow \\ |dr| &= \sqrt{dr \cdot dr} = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1 \Rightarrow \\ dr &= d\hat{r} \end{aligned} \quad (33)$$

$$\begin{aligned} d\theta &= |d\theta|d\hat{\theta} = \left| -\frac{\sin(\theta)}{r}d\hat{x} + \frac{\cos(\theta)}{r}d\hat{y} \right| d\hat{\theta} \Rightarrow \\ |d\theta| &= \sqrt{d\theta \cdot d\theta} = \sqrt{\frac{1}{r^2}(\sin^2(\theta) + \cos^2(\theta))} = \frac{1}{r} \Rightarrow \\ d\theta &= \frac{1}{r}d\hat{\theta} \Rightarrow \end{aligned} \quad (34)$$

Note: $\begin{bmatrix} dx \\ dy \end{bmatrix}$ are row vectors

Putting the components and basis vectors into equation (26) $\Rightarrow$

$$(\bar{\nabla}\varphi)_i \bar{W} = \begin{bmatrix} \frac{\partial \varphi}{\partial r} & \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \end{bmatrix} \begin{bmatrix} d\hat{r} \\ d\hat{\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{bmatrix} \begin{bmatrix} d\hat{x} \\ d\hat{y} \end{bmatrix} = (\nabla\varphi)_i W \quad (35)$$

Equation (35) is the gradient of the polar system.

Gradient as a Vector in Polar Coordinates

Convert the gradient from a one form basis to a vector basis $\Rightarrow$

$$\mathbf{u} = \bar{u}_i \bar{W} = u_i W \quad (26)$$

For $\mathbf{u} = \nabla \varphi \Rightarrow$

$$\nabla \varphi = (\bar{\nabla} \varphi)_i \bar{W} = (\nabla \varphi)_i W \quad (36)$$

Also, the equations for raising and lowering a vector and one form are given below - See²

$$\begin{aligned} u_i &= u^i G \\ u^i &= u_i G^{-1} \end{aligned} \quad (37)$$

Now the basis vectors are the rows of both the E and W matrix, so

$$E = WG \quad (38)$$

$$W = EG^{-1} \quad (39)$$

$$\bar{W} = B^T I = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix}$$

$$\bar{E} = AI = A$$

$$\bar{G} = \bar{E} \bar{E}^T = AA^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad (40)$$

$$\bar{G}^{-1} = WW^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\frac{\sin(\theta)}{r} \\ \sin(\theta) & \frac{\cos(\theta)}{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix} \quad (41)$$

² [Coordinates Summary p. 5-6](#)

$$\bar{E} = \bar{W}\bar{G} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \quad (42)$$

Equation (42) correctly transforms $\bar{W}$ to $\bar{E}$

Also

$$\bar{G} = AGA^T \quad (43)$$

$$\bar{G}^{-1} = B^T G^{-1} B \quad (44)$$

The right side of equation (26) $\Rightarrow$

$$u_i W = u^i E \Rightarrow (\nabla\varphi)_i W = (\nabla\varphi)^i E \quad (45)$$

In Cartesian Coordinates, E , W , G , and G^{-1} all equal the identity matrix $I \Rightarrow$

$$u^i = u_i \Rightarrow$$

Equation (45) $\Rightarrow$

$$(\nabla\varphi)_i dx^i = (\nabla\varphi)^i \frac{\partial}{\partial x^i} = \begin{bmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \end{bmatrix} \quad (46)$$

The left side of equation (26) $\Rightarrow$

$$\begin{aligned} \bar{u}_i \bar{W} &= \bar{u}^i \bar{E} = \bar{u}_i \bar{G}^{-1} \bar{E} = (\bar{\nabla}\varphi)_i \bar{G}^{-1} \bar{E} = \begin{bmatrix} \frac{\partial\varphi}{\partial r} & \frac{1}{r} \frac{\partial\varphi}{\partial\theta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial\varphi}{\partial r} & \frac{1}{r^3} \frac{\partial\varphi}{\partial\theta} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial\varphi}{\partial r} & \frac{1}{r^3} \frac{\partial\varphi}{\partial\theta} \end{bmatrix} \begin{bmatrix} \cos(\theta) \frac{\partial}{\partial x} + \sin(\theta) \frac{\partial}{\partial y} \\ -r\sin(\theta) \frac{\partial}{\partial x} + r\cos(\theta) \frac{\partial}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial\varphi}{\partial r} & \frac{1}{r^3} \frac{\partial\varphi}{\partial\theta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ r \frac{\partial}{\partial\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial\varphi}{\partial r} & \frac{1}{r^2} \frac{\partial\varphi}{\partial\theta} \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \end{bmatrix} \end{aligned} \quad (47)$$

Equating equation (47) with equation (46) $\Rightarrow$

$$(\nabla\varphi)^i \frac{\partial}{\partial \bar{x}^i} = \left[\frac{\partial\varphi}{\partial r} \quad \frac{1}{r^2} \frac{\partial\varphi}{\partial \theta} \right] \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \end{bmatrix} = \left[\frac{\partial\varphi}{\partial x} \quad \frac{\partial\varphi}{\partial y} \right] \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \end{bmatrix} = (\nabla\varphi)^i \frac{\partial}{\partial x^i} = \left[\frac{\partial\varphi}{\partial x} \quad \frac{\partial\varphi}{\partial y} \right] \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \end{bmatrix} \quad (48)$$

Note: $(\nabla\varphi)^i = (d\varphi)^i$

Directional Derivative

The derivation of the directional derivative will follow – see ³. Consider a curve $C(\lambda)$ on a manifold and a point P on $C(\lambda)$. Define λ as a parameter along the curve, set $C(0) = P$ and let f be a function.

Define a vector on a two-dimensional surface.

$$\mathbf{v} = v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y} = [v^x \quad v^y] \quad (49)$$

Define a section of curve that intersects $\mathbf{v}$ as shown in equation (50) $\Rightarrow$

$$C(\lambda) = v^x \lambda \frac{\partial}{\partial x} + v^y \lambda \frac{\partial}{\partial y} = [v^x \lambda \quad v^y \lambda] = [x \quad y] \quad (50)$$

where

$$\begin{aligned} x &= \lambda v^x \\ y &= \lambda v^y \end{aligned}$$

Given the above parameters,

$$C(0) = [0 \quad 0] \text{ and}$$

$$C(1) = [v^x \quad v^y]$$

Treat f as a function of n variables – $n = 2$ in the two-dimensional case

$$f(C(\lambda)) = f([v^x \lambda \quad v^y \lambda]) = f(x, y)$$

Expand $f(C(\lambda))$ in a two-dimensional Taylor Series to first order $\Rightarrow$

$$f(C(\lambda)) = f(x, y) = f(0, 0) + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + \dots = f(0, 0) + v^x \lambda \frac{\partial f}{\partial x} + v^y \lambda \frac{\partial f}{\partial y} \quad (51)$$

³ Sadri Hassani, Foundations of Mathematical Physics, Allyn and Bacon, 1991, pp. 257-258

Now, find the derivative at point P along $C(\lambda)$ through vector $\mathbf{v}$.

$$\frac{df(C(\lambda))}{d\lambda} = \lim_{\lambda \rightarrow 0} \frac{f(0,0) + v^x \lambda \frac{\partial f}{\partial x} + v^y \lambda \frac{\partial f}{\partial y}}{\lambda} \quad (52)$$

Because,

$$\lim_{\lambda \rightarrow 0} \frac{\lambda}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{1}{1} = 1$$

Equation (52) $\Rightarrow$

$$\frac{df(C(\lambda))}{d\lambda} = v^x \frac{\partial f}{\partial x} + v^y \frac{\partial f}{\partial y} = \left(v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y} \right) f = v^i \frac{\partial f}{\partial x^i} \quad (53)$$

Equation (53) can be converted using the metric to $v_i(df)^i \Rightarrow$

$$D_v f = \nabla f \cdot \mathbf{v} = v^i \frac{\partial f}{\partial x^i} = v_i(df)^i \quad (54)$$

Compute Transformation of a Partial Derivative – $\partial v^i / \partial q^i$ – using Standard Index Equations

The derivation of the coordinate transform of a partial derivative using standard index equations will follow - see.⁴

$$\bar{v}^i = \frac{\partial \bar{q}^i}{\partial q^j} v^j \quad (55)$$

$$\frac{\partial \bar{v}^i}{\partial \bar{q}^k} = \frac{\partial}{\partial \bar{q}^k} \left(\frac{\partial \bar{q}^i}{\partial q^j} v^j \right) \quad (56)$$

The right side of equation (56) takes a derivative of the expression in parentheses with respect to $\bar{q}^k$. This can be changed to take the derivative with respect to q^k by using equation (57).

⁴ David McMahon, Relativity Demystified, McGraw-Hill, 2006, pp. 65 - 66

$$\frac{\partial q^l}{\partial q^m} = \delta_m^l = \begin{cases} 1, l = m \\ 0, l \neq m \end{cases} \quad (57)$$

Equation (57) can be rewritten as

$$\frac{\partial q^l}{\partial q^l} = 1 \quad (58)$$

Using equation (58) in equation (56) $\Rightarrow$

$$\begin{aligned} \frac{\partial \bar{v}^i}{\partial \bar{q}^k} &= \frac{\partial}{\partial \bar{q}^k} \left(\frac{\partial q^l}{\partial q^l} \frac{\partial \bar{q}^i}{\partial q^j} v^j \right) = \frac{\partial q^l}{\partial \bar{q}^k} \frac{\partial}{\partial q^l} \left(\frac{\partial \bar{q}^i}{\partial q^j} v^j \right) = \frac{\partial q^l}{\partial \bar{q}^k} \left[\frac{\partial^2 \bar{q}^i}{\partial q^l \partial q^j} v^j + \frac{\partial \bar{q}^i}{\partial q^j} \frac{\partial v^j}{\partial q^l} \right] \\ &= \frac{\partial q^l}{\partial \bar{q}^k} \frac{\partial^2 \bar{q}^i}{\partial q^l \partial q^j} v^j + \frac{\partial q^l}{\partial \bar{q}^k} \frac{\partial \bar{q}^i}{\partial q^j} \frac{\partial v^j}{\partial q^l} \end{aligned} \quad (59)$$

Equation (59) is the standard equation for the transformation of a partial derivative.

Note: the partial derivative is not a tensor. To be a tensor, the equation would have to be of the form.

$$\frac{\partial \bar{v}^i}{\partial \bar{q}^k} = T \frac{\partial v^j}{\partial q^l} \quad (60)$$

where

$$T = \frac{\partial q^l}{\partial \bar{q}^k} \frac{\partial \bar{q}^i}{\partial q^j}$$

Compute Transformation of a Partial Derivative – $\partial v^i / \partial q^l$ – Using Matrix Approach

The relations in equation (61) are used to convert equation (59) using the matrix approach $\Rightarrow$

$$A = \frac{\partial \mathbf{q}}{\partial \bar{\mathbf{q}}} = \frac{\partial q^l}{\partial \bar{q}^k}$$

$$B = \frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{q}} = \frac{\partial \bar{q}^i}{\partial q^j}$$

$$\bar{\mathbf{v}}^i = \mathbf{v}^i B \quad (61)$$

Using the expression for the A matrix from equations (61) see ⁵, equation (59) $\Rightarrow$

$$\frac{\partial \bar{v}^i}{\partial \bar{q}^k} = \frac{\partial}{\partial \bar{q}^k} \left(\frac{\partial q^l}{\partial q^l} \frac{\partial \bar{q}^i}{\partial q^j} v^j \right) = \frac{\partial q^l}{\partial \bar{q}^k} \frac{\partial}{\partial q^l} \left(\frac{\partial \bar{q}^i}{\partial q^j} v^j \right) \Rightarrow$$
(62)

Now finish putting equation (62) into matrix format.

$$\bar{v}^i = v^j \frac{\partial \bar{q}^i}{\partial q^j} \Rightarrow \bar{v}^i = v^i B$$

$$\frac{\partial \bar{v}^i}{\partial \bar{q}^k} = \frac{\partial}{\partial \bar{q}^k} [v^i B] = \frac{\partial q^l}{\partial \bar{q}^k} \left[\frac{\partial v^i}{\partial q^l} B + v^i \frac{\partial B}{\partial q^l} \right]$$
(63)

But in this case k is a fixed index, so

$$\frac{\partial \bar{v}^i}{\partial \bar{q}^k} = \frac{\partial q^l}{\partial \bar{q}^k} \left[\frac{\partial v^i}{\partial q^l} B + v^i \frac{\partial B}{\partial q^l} \right] = \frac{\partial q^1}{\partial \bar{q}^k} \left[\frac{\partial v^i}{\partial q^1} B + v^i \frac{\partial B}{\partial q^1} \right] + \dots + \frac{\partial q^n}{\partial \bar{q}^k} \left[\frac{\partial v^i}{\partial q^n} B + v^i \frac{\partial B}{\partial q^n} \right]$$
(64)

The A matrix is defined below.

$$A = \begin{bmatrix} \frac{\partial \mathbf{q}}{\partial \bar{\mathbf{q}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial q^1}{\partial \bar{q}^1} & \dots & \frac{\partial q^n}{\partial \bar{q}^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial q^1}{\partial \bar{q}^n} & \dots & \frac{\partial q^n}{\partial \bar{q}^n} \end{bmatrix}$$
(65)

Equations (65) is the definition of differentiating a vector $\mathbf{q}$ with respect to $\bar{\mathbf{q}}$.

Notice, in equation (65), that each row of the matrix is the derivative of the vector with respect to a single variable $\bar{q}^k$. Using this same logic $\Rightarrow$

$$\frac{\partial B}{\partial \bar{q}^k} = \begin{bmatrix} \frac{\partial B}{\partial \bar{q}^1} \\ \vdots \\ \frac{\partial B}{\partial \bar{q}^n} \end{bmatrix}$$
(66)

Picture equation (66) as a vector of matrices

⁵ [CoordinatesSummary-pages-8-11](#)

$$v^i \frac{\partial B}{\partial \bar{q}^k} = \begin{bmatrix} v^i \frac{\partial B}{\partial \bar{q}^1} \\ \vdots \\ v^i \frac{\partial B}{\partial \bar{q}^n} \end{bmatrix} \quad (67)$$

$$v^i \frac{\partial B}{\partial \bar{q}^j} \equiv \text{a vector}$$

Picture equation (67) as a vector of vectors or a matrix. It could also be pictured as a set of equations where each equation uses a derivative with respect to a fixed index - $\bar{q}^p$ as was used in previous writeups.

The A matrix writes the primed basis vectors as a function of the non-primed basis vectors. Similarly, the B matrix writes the primed vector components in terms of the non-primed vector components. The coordinate relations can map between the primed and non-primed coordinate variables.

Equation (64) can be put into a form using the A matrix as shown in equation (68).

First declare the vector $\mathbf{u}$

$$\begin{aligned} \mathbf{u} &= \left[\left[\frac{\partial v^i}{\partial q^1} B + v^i \frac{\partial B}{\partial q^1} \right] \quad \left[\frac{\partial v^i}{\partial q^2} B + v^i \frac{\partial B}{\partial q^2} \right] \quad \dots \quad \left[\frac{\partial v^i}{\partial q^n} B + v^i \frac{\partial B}{\partial q^n} \right] \right] \\ \left[\frac{\partial \bar{v}^i}{\partial \bar{q}^k} \right]^T &= A \cdot \mathbf{u}^T \Rightarrow \frac{\partial \bar{v}^i}{\partial \bar{q}^k} = [A \cdot \mathbf{u}^T]^T = \mathbf{u} \cdot A^T \\ &= \left[\frac{\partial v^i}{\partial q^1} B + v^i \frac{\partial B}{\partial q^1} \right] A_{k1} + \dots + \left[\frac{\partial v^i}{\partial q^n} B + v^i \frac{\partial B}{\partial q^n} \right] A_{kn} \end{aligned} \quad (68)$$

In equation (68), $\mathbf{u}$ is a block vector whose elements are vectors. The ‘ $\cdot$ ’ is used to indicate that it’s a block vector operation as indicated in equation (68). The indices are reversed because $[A^T]_{ij} = A_{ji}$

Parabolic Coordinates

Now we will see an example of how partial derivatives transform. We pick parabolic coordinates because the derivatives with respect to the basis vectors do not vanish. The coordinate equations are given by equation (69).

$$x = \sigma\tau$$

$$y = \frac{1}{2}(\tau^2 - \sigma^2) \quad (69)$$

Now the A matrix is given by equations (70).

$$\begin{aligned} \frac{\partial}{\partial \sigma} &= \frac{\partial x}{\partial \sigma} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \sigma} \frac{\partial}{\partial y} = \tau \frac{\partial}{\partial x} - \sigma \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \tau} &= \frac{\partial x}{\partial \tau} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \tau} \frac{\partial}{\partial y} = \sigma \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial y} \end{aligned} \quad (70)$$

Equation (70) $\Rightarrow$

$$A = \begin{bmatrix} \tau & -\sigma \\ \sigma & \tau \end{bmatrix} \quad (71)$$

where the basis vectors are the rows of A .

$$\sigma = [\tau \quad -\sigma]$$

$$\tau = [\sigma \quad \tau]$$

Inverse Parabolic Coordinates

Using Maxima⁶ to solve equations (69) for σ and $\tau \Rightarrow 4$ roots

1.

$$\begin{aligned} \sigma &= -\sqrt{\sqrt{x^2 + y^2} - y} \\ \tau &= -\frac{\sqrt{\sqrt{x^2 + y^2} - y}(\sqrt{x^2 + y^2} + y)}{x} \end{aligned} \quad (72)$$

2.

$$\sigma = \sqrt{\sqrt{x^2 + y^2} - y}$$

⁶ <https://maxima - computer algebra system>

$$\tau = \frac{\sqrt{\sqrt{x^2 + y^2} - y(\sqrt{x^2 + y^2}) + y}}{x} \quad (73)$$

3.

$$\sigma = -\sqrt{-\sqrt{x^2 + y^2} - y}$$

$$\tau = \frac{\sqrt{-\sqrt{x^2 + y^2} - y(\sqrt{x^2 + y^2} - y)}}{x} \quad (74)$$

4.

$$\sigma = \sqrt{-\sqrt{x^2 + y^2} - y}$$

$$\tau = -\frac{\sqrt{-\sqrt{x^2 + y^2} - y(\sqrt{x^2 + y^2} - y)}}{x} \quad (75)$$

Figure 5 shows a parabolic coordinate grid.

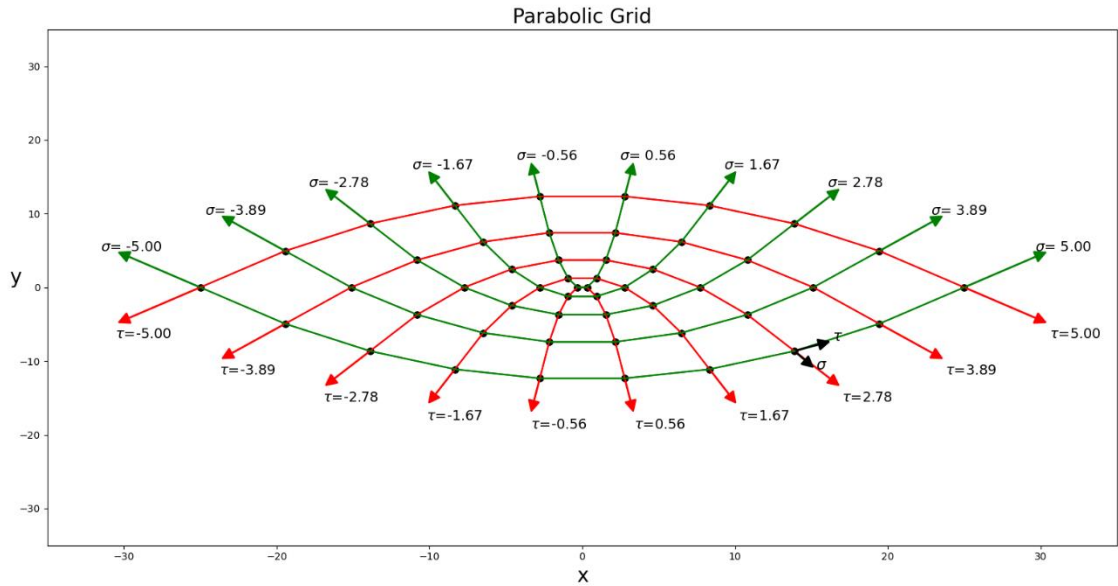


Figure 5

Several things in Figure 5 need to be discussed. First the range of both σ and τ is $[-5, 5]$. A curve of constant $(-\sigma)$ or $(-\tau)$ is the same curve but progresses in the opposite direction as curves of constant σ or τ . The grid can't be used in the same way an xy grid is used to locate unique points. For example, the $\sigma = 0.167$ intersects $\tau = 2.78$ in two places. The grid lines do

give the correct direction of the basis vectors. The basis vectors (scaled) are shown at point $(\sigma = 5, \tau = 2.78)$ in Figure 5. The σ basis vector points along the constant τ grid line in the positive direction. The τ basis vector points along the constant σ grid line in the positive direction.

Compute $\partial \bar{v}^i / \partial \bar{q}^i$ in Parabolic Coordinates

It's easier to use (σ, τ) coordinates for the computation.

$$B = A^{-1} = \frac{1}{\tau^2 + \sigma^2} \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} \quad (76)$$

$$\frac{\partial B}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\left(\frac{1}{\tau^2 + \sigma^2} \right) \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} \right) = \frac{1}{(\tau^2 + \sigma^2)^2} \begin{bmatrix} -2\sigma\tau & \tau^2 - \sigma^2 \\ -(\tau^2 - \sigma^2) & -2\sigma\tau \end{bmatrix} \quad (77)$$

$$\frac{\partial B}{\partial \tau} = \frac{\partial}{\partial \tau} \left(\left(\frac{1}{\tau^2 + \sigma^2} \right) \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} \right) = \frac{1}{(\tau^2 + \sigma^2)^2} \begin{bmatrix} -(\tau^2 - \sigma^2) & -2\tau\sigma \\ 2\tau\sigma & -(\tau^2 - \sigma^2) \end{bmatrix} \quad (78)$$

Compute Vector with Respect to σ

Plugging (77) into equation (65)

$$\frac{\partial \bar{v}^i}{\partial \sigma} = \frac{\partial v^i}{\partial \sigma} \frac{1}{(\tau^2 + \sigma^2)} \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} + \frac{v^i}{(\tau^2 + \sigma^2)^2} \begin{bmatrix} -2\sigma\tau & \tau^2 - \sigma^2 \\ -(\tau^2 - \sigma^2) & -2\sigma\tau \end{bmatrix} \quad (79)$$

Define a vector field using Cartesian basis vectors with (σ, τ) coordinates. Follow the numbered steps below at each field point to be clear as to what this means $\Rightarrow$

- 1.) Start out picking specific (σ, τ) coordinate values
- 2.) Use coordinate relations – equations (69) – to convert the (σ, τ) coordinates to (x, y) coordinates
- 3.) The (σ, τ) and (x, y) coordinates refer to the same geometric point
- 4.) σ and τ are numbers, so use these numbers to define a vector field using the (x, y) basis vectors e_x and e_y .
- 5.) The A and B matrix are formed using the same (σ, τ) numbers.

$$v^i = \sigma e_x + \tau e_y = [\sigma \quad \tau] \quad (80)$$

First Compute the derivative in the (x, y) system and putting into equation (79)

$$\frac{\partial v^i}{\partial \sigma} = [1 \quad 0] \quad (81)$$

Putting equation (81) into equation (79) $\Rightarrow$

$$\begin{aligned} \frac{\partial \bar{v}^i}{\partial \sigma} &= \frac{[1 \quad 0]}{(\tau^2 + \sigma^2)} \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} + \frac{[\sigma \quad \tau]}{(\tau^2 + \sigma^2)^2} \begin{bmatrix} -2\sigma\tau & \tau^2 - \sigma^2 \\ -(\tau^2 - \sigma^2) & -2\sigma\tau \end{bmatrix} = \\ &= \frac{[\tau \quad \sigma]}{(\tau^2 + \sigma^2)} + \frac{[-2\sigma^2\tau - \tau^3 + \tau\sigma^2 \quad \sigma\tau^2 - \sigma^3 - 2\sigma\tau^2]}{(\tau^2 + \sigma^2)^2} \end{aligned} \quad (82)$$

Multiply the left side of equation (82) by $(\tau^2 + \sigma^2)$ to give a common denominator of $(\tau^2 + \sigma^2)^2 \Rightarrow$

$$\begin{aligned} \frac{\partial \bar{v}^i}{\partial \sigma} &= \frac{[\tau^3 + \tau\sigma^2 \quad \sigma\tau^2 + \sigma^3]}{(\tau^2 + \sigma^2)^2} + \frac{[-2\sigma^2\tau - \tau^3 + \tau\sigma^2 \quad \sigma\tau^2 - \sigma^3 - 2\sigma\tau^2]}{(\tau^2 + \sigma^2)^2} \\ &= \frac{[\tau^3 + \tau\sigma^2 - \tau\sigma^2 - \tau^3 \quad \sigma\tau^2 + \sigma^3 - \sigma\tau^2 - \sigma^3]}{(\tau^2 + \sigma^2)^2} \\ &= [0 \quad 0] \end{aligned} \quad (83)$$

Now transform the vector to the (σ, τ) system first, and then taking the derivative as a comparison.

Using B from equation (76) $\Rightarrow$

$$\bar{v}^i = v^i B = \frac{[\sigma \quad \tau]}{\tau^2 + \sigma^2} \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} = \frac{[\sigma\tau - \tau\sigma \quad \sigma^2 + \tau^2]}{\tau^2 + \sigma^2} = [0 \quad 1] \quad (84)$$

$$\frac{\partial \bar{v}^i}{\partial \sigma} = \frac{\partial [0 \quad 1]}{\partial \sigma} = [0 \quad 0] \quad (85)$$

Equation (85) agrees with equation (83) as expected.

Compute Vector with Respect to τ

$$\frac{\partial v^i}{\partial \tau} = \frac{\partial [\sigma \quad \tau]}{\partial \tau} = [0 \quad 1] \quad (86)$$

Putting equations (80), (78), and (76) into equation (65) $\Rightarrow$

$$\begin{aligned} \frac{\partial \bar{v}^i}{\partial \tau} &= \frac{[0 \quad 1]}{(\tau^2 + \sigma^2)} \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} + \frac{[\sigma \quad \tau]}{(\tau^2 + \sigma^2)^2} \begin{bmatrix} -(\tau^2 - \sigma^2) & -2\tau\sigma \\ 2\tau\sigma & -(\tau^2 - \sigma^2) \end{bmatrix} = \\ &= \frac{[-\sigma \quad \tau]}{(\tau^2 + \sigma^2)} + \frac{[-\sigma(\tau^2 - \sigma^2) + 2\tau^2\sigma \quad -2\tau\sigma^2 - \tau(\tau^2 - \sigma^2)]}{(\tau^2 + \sigma^2)^2} \end{aligned} \quad (87)$$

Multiplying the left side of equation (87) by $(\tau^2 + \sigma^2)$ to get a common denominator of $(\tau^2 + \sigma^2)^2 \Rightarrow$

$$\begin{aligned} &\frac{[-\sigma\tau^2 - \sigma^3 \quad \tau^3 + \tau\sigma^2]}{(\tau^2 + \sigma^2)^2} + \frac{[-\sigma\tau^2 + \sigma^3 + 2\tau^2\sigma \quad -2\tau\sigma^2 - \tau^3 + \tau\sigma^2]}{(\tau^2 + \sigma^2)^2} = \\ &\frac{[-\sigma\tau^2 - \sigma^3 - \sigma\tau^2 + \sigma^3 + 2\tau^2\sigma \quad \tau^3 + \tau\sigma^2 - 2\tau\sigma^2 - \tau^3 + \tau\sigma^2]}{(\tau^2 + \sigma^2)^2} = [0 \quad 0] \end{aligned} \quad (88)$$

Now transform v^i to $\bar{v}^i$, and then take the derivative.

Using B from equation (76) $\Rightarrow$

$$\begin{aligned} \bar{v}^i &= v^i B = \frac{[\sigma \quad \tau]}{\tau^2 + \sigma^2} \begin{bmatrix} \tau & \sigma \\ -\sigma & \tau \end{bmatrix} = \frac{[\sigma\tau - \tau\sigma \quad \sigma^2 + \tau^2]}{\tau^2 + \sigma^2} = [0 \quad 1] \Rightarrow \\ \frac{\partial \bar{v}^i}{\partial \tau} &= \frac{\partial [0 \quad 1]}{\partial \tau} = [0 \quad 0] \end{aligned} \quad (89)$$

Equation (89) agrees with equation (88) as expected.