# Rank One Update And the Google Matrix

**by**

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There are two different ways to perform matrix multiplication. The first uses a dot product formulation as shown in equation (1)

$$(A \bullet B)_{i,j} = \sum_k a_{i,k} b_{k,j} \qquad (1)$$

The sum is over the variable $k = 1, \cdots, N$

This is the standard matrix multiplication for example with i=1 and j=1,

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \end{bmatrix} \bullet \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \end{bmatrix} = a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} \qquad (2)$$

Notice that in equation (1) the indices i and j are fixed variables; i.e. we specify these on the left side of the equation when we pick the i-j element. There is another way to look at the equation; that is keep the indices i and j free as shown in equation (3).

$$(A \bullet B) = \sum_k a_{i,k} b_{k,j} \qquad (3)$$

where $k = 1, \cdots N$.

In this case the multiplication for each k is a matrix. This formula is the called the outer product formulation.

$$\begin{bmatrix} a_{1,1}b_{1,1} & \cdots & a_{1,1}b_{1,N} \\ a_{2,1}b_{1,1} & \cdots & a_{2,1}b_{1,N} \\ a_{N,1}b_{1,1} & \cdots & a_{N,1}b_{1,N} \end{bmatrix} + \begin{bmatrix} a_{1,2}b_{2,1} & \cdots & a_{1,2}b_{2,N} \\ a_{2,2}b_{2,1} & \cdots & a_{2,2}b_{2,N} \\ a_{N,2}b_{2,1} & \cdots & a_{N,2}b_{2,N} \end{bmatrix} + \cdots + \begin{bmatrix} a_{1,N}b_{N,1} & \cdots & a_{1,N}b_{N,N} \\ a_{2,N}b_{N,1} & \cdots & a_{2,N}b_{N,N} \\ a_{N,N}b_{N,1} & \cdots & a_{N,N}b_{N,N} \end{bmatrix} =$$
$$\begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + \cdots + a_{1,N}b_{N,1} & \cdots & a_{1,1}b_{1,N} + a_{1,2}b_{2,N} + \cdots + a_{1,N}b_{N,N} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + \cdots + a_{2,N}b_{N,1} & \cdots & a_{2,1}b_{1,N} + a_{2,2}b_{2,N} + \cdots + a_{2,N}b_{N,N} \\ a_{N,1}b_{1,1} + a_{N,2}b_{2,1} + \cdots + a_{N,N}b_{N,1} & \cdots & a_{N,1}b_{1,N} + a_{N,2}b_{2,N} + \cdots + a_{N,N}b_{N,N} \end{bmatrix} \qquad (4)$$

The outer product operation is defined below

$$\vec{a} \otimes \vec{b} = a_i b_j = \begin{bmatrix} a_0 b_0 & \cdots & a_0 b_N \\ \vdots & & \\ a_N b_0 & \cdots & a_N b_N \end{bmatrix} = \vec{a}^T \vec{b} \qquad (5)$$

Remember: $\vec{a}^T$ is a column vector and $\vec{b}$ is a row vector

Equation (5) in outer product notation

$$AB = \sum_i \left( \vec{a}_i \otimes \vec{b}_i \right) \qquad (6)^1$$

A Rank One update is of the form

$$A + \vec{a}^T \vec{b} \qquad (7)$$

The Google matrix, G, is defined in equation (7)

$$G = \alpha S + \frac{(1-\alpha)}{n} \vec{e} \bullet \vec{e}^T \qquad (8)^2$$

where

$\alpha$ $\equiv$scalar number between 0 and 1
$n \equiv$number of pages in search space,
$e^T$ $\equiv$ is a row vector of elements (1/n)
$S$ $\equiv$ defined by equation (9)

$$S = H + \frac{\vec{a}}{n} \vec{e}^T \qquad (9)$$

where
$a$ is a row vector of dangling links; 0 if the link on the i[th] node is dangling (i.e. a node that is does not navigate to another page on the web such as a file) and is 1 otherwise.
$H$ is the hyperlink matrix which represents a graph of how web pages are connected through links. This write up is not intended to go into detail about Google page rank. The main point here is that equations (8) and (9) both use the Rank One Update (7).

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[1]Gerald Bierman, “Fatorizing Methods for Discrete Sequential Estimation”, Accademic Press, 1977, page 25
[2] Langville and Meyer, “Google’s Page Rank and Beyond”, Princeton University Press, 2006, pp 37-38