Tensor Transformations

By

Al Bernstein

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http://www.metricmath.com

al@metricmath.com

Introduction

A general tensor is defined as shown in equation (1).

$$\boldsymbol{T} = T^{\alpha\beta\cdots}_{\gamma\delta\cdots}\boldsymbol{\omega}^{\gamma} \otimes \boldsymbol{\omega}^{\delta} \otimes \boldsymbol{e}_{\alpha} \otimes \boldsymbol{e}_{\beta} \cdots \tag{1}$$

A few points about equation (1).

1.) A tensor is invariant with a change in coordinates and is denoted as $\boldsymbol{T}$

2.) $\otimes$ is a tensor product operator

3.) The basis set is $\boldsymbol{\omega}^{\gamma} \otimes \boldsymbol{\omega}^{\delta} \otimes \boldsymbol{e}_{\alpha} \otimes \boldsymbol{e}_{\beta} \cdots$

4.) The rank of the tensor is the number of indices – i.e. the number of basis vectors and reciprocal basis vectors.

5.) $T^{\alpha\beta\cdots}_{\gamma\delta\cdots}$ are the coefficients of the basis set.

6.) Einstein summation convention applies with an implied sum over the same indices with one as a subscript and the other as a superscript.

The tensor product of $Object_1 \otimes Object_2$ is computed by multiplying each element of $Object_1$ by $Object_2$.

The tensor product of two vectors is defined as follows:

$$\boldsymbol{e_1} = [e_{11} \quad \cdots \quad e_{1n}]$$

$$\boldsymbol{e_2} = [e_{11} \quad \cdots \quad e_{1n}]$$

$$\boldsymbol{e_1} \otimes \boldsymbol{e_2} = e_{1i}e_{2j} = \{e_{11}e_{21} \quad \cdots \quad e_{11}e_{2n} \quad e_{1n}e_{21} \cdots \quad e_{1n}e_{2n}\} = \begin{bmatrix} e_{11}e_{21} & \cdots & e_{11}e_{2n} \\ \vdots & \ddots & \vdots \\ e_{1n}e_{21} & \cdots & e_{1n}e_{2n} \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} \\ \vdots \\ e_{1n} \end{bmatrix} \begin{bmatrix} e_{21} & \cdots & e_{2n} \end{bmatrix} = \boldsymbol{e_1^T e_2} \tag{2}$$

Equation (2) is a set of pairs generated from the tensor product. A second order tensor can be computed as a sum of coefficients multiplied by tensor products of the basis vectors – basis set. An example of the expansion of a second order tensor is shown in equation (3) ⇒

$$\boldsymbol{T} = T^{ij}\boldsymbol{e_i} \otimes \boldsymbol{e_j} = T^{11}\boldsymbol{e_1} \otimes \boldsymbol{e_1} + T^{12}\boldsymbol{e_1} \otimes \boldsymbol{e_2} + T^{21}\boldsymbol{e_2} \otimes \boldsymbol{e_1} + T^{22}\boldsymbol{e_2} \otimes \boldsymbol{e_2} \tag{3}$$

where

$i = 1,2$ and $j = 1,2$

Equation (4) shows the tensor product for three basis vectors.

$$\boldsymbol{e_1} \otimes \boldsymbol{e_2} \otimes \boldsymbol{e_3} = e_{1i}e_{2j}e_{3k} = \begin{bmatrix} \begin{bmatrix} e_{11}e_{21} & \cdots & e_{11}e_{2n} \\ \vdots & \ddots & \vdots \\ e_{1n}e_{21} & \cdots & e_{1n}e_{2n} \end{bmatrix} e_{31} & \cdots & \begin{bmatrix} e_{11}e_{21} & \cdots & e_{11}e_{2n} \\ \vdots & \ddots & \vdots \\ e_{1n}e_{21} & \cdots & e_{1n}e_{2n} \end{bmatrix} e_{3n} \end{bmatrix} \tag{4}$$

Equation (4) can be viewed as a block vector – a vector of matrices - and is convenient for doing computations. A third order tensor with contravariant components is shown in equation (5).

$$\boldsymbol{T} = T^{ijk}\boldsymbol{e_i} \otimes \boldsymbol{e_j} \otimes \boldsymbol{e_k} \tag{5}$$

Using Einstein's summation convention, equation (5) is the sum of the tensor coefficients multiplied by each element of the basis set – tensor products of three basis vectors.

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Tensor Transformations

A general tensor can be transformed by transforming the basis vectors that make up the basis set and its components to keep the tensor invariant. A tensor is invariant under a coordinate transform because it is designed to be the same mathematical object independent of coordinates.

Equation (6) expresses the invariance of a tensor under a coordinate transformation.

$$\boldsymbol{T} = \boldsymbol{T} = T^{\alpha\beta\cdots}_{\gamma\delta\cdots}\boldsymbol{\omega^\gamma} \otimes \boldsymbol{\omega^\delta} \otimes \boldsymbol{e_\alpha} \otimes \boldsymbol{e_\beta} \cdots = T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots}\boldsymbol{\omega^{\bar{\gamma}}} \otimes \boldsymbol{\omega^{\bar{\delta}}} \otimes \boldsymbol{e_{\bar{\alpha}}} \otimes \boldsymbol{e_{\bar{\beta}}} \cdots \tag{6}$$

There are two types of components in a general tensor basis – basis vectors - $\boldsymbol{e_\alpha}$ – and reciprocal basis vectors - $\boldsymbol{e^\gamma}$.

Equation (7) shows how each of these basis vectors transform – see[1]

[1] Coordinates Summary pp. 9-11

$$e_{\bar{\alpha}} = Ae_{\alpha} = \frac{\partial q^{\alpha}}{\partial q^{\bar{\alpha}}} e_{\alpha}$$

$$\omega^{\bar{\gamma}} = B^T \omega^{\gamma} = \frac{\partial q^{\bar{\gamma}}}{\partial q^{\gamma}} \omega^{\gamma} \tag{7}$$

Equation (8) shows a tensor in the primed coordinate system written in terms of unprimed basis vectors and transformation matrices.

$$T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} \omega^{\bar{\gamma}} \otimes \omega^{\bar{\delta}} \otimes e_{\bar{\alpha}} \otimes e_{\bar{\beta}} \cdots = T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} (B^T \omega^{\gamma}) \otimes \left(B^T \omega^{\delta}\right) \otimes (Ae_{\alpha}) \otimes \left(Ae_{\beta}\right) \otimes \cdots$$

$$= T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} [B^T \omega^{\gamma}]^{\bar{\gamma}} \left[B^T \omega^{\delta}\right]^{\bar{\delta}} [Ae_{\alpha}]_{\bar{\alpha}} \left[Ae_{\beta}\right]_{\bar{\beta}} \cdots = T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} \frac{\partial q^{\bar{\gamma}}}{\partial q^{\gamma}} \omega^{\gamma} \frac{\partial q^{\bar{\delta}}}{\partial q^{\delta}} \omega^{\delta} \frac{\partial q^{\alpha}}{\partial q^{\bar{\alpha}}} e_{\alpha} \frac{\partial q^{\beta}}{\partial q^{\bar{\beta}}} e_{\beta} \cdots \tag{8}$$

Because the terms in equation (8) are independent of each other and are connected through Einstein's summation convention, the terms in equation (8) can be regrouped without affecting the equation. Equation (9) shows equation (6) where the basis vectors are written in terms of the unprimed basis vectors.

$$T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} \omega^{\bar{\gamma}} \otimes \omega^{\bar{\delta}} \otimes e_{\bar{\alpha}} \otimes e_{\bar{\beta}} \cdots = T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} \frac{\partial q^{\bar{\gamma}}}{\partial q^{\gamma}} \frac{\partial q^{\bar{\delta}}}{\partial q^{\delta}} \frac{\partial q^{\alpha}}{\partial q^{\bar{\alpha}}} \frac{\partial q^{\beta}}{\partial q^{\bar{\beta}}} \cdots \omega^{\gamma} \omega^{\delta} e_{\beta} e_{\alpha} \cdots$$

$$= T^{\alpha\beta\cdots}_{\gamma\delta\cdots} \omega^{\gamma} \omega^{\delta} e_{\beta} e_{\alpha} \cdots \tag{9}$$

Equation (9) now has the same basis set on each side, so we can equate the coefficients and transformations with each other. ⇒

$$T^{\alpha\beta\cdots}_{\gamma\delta\cdots} = T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} \frac{\partial q^{\bar{\gamma}}}{\partial q^{\gamma}} \frac{\partial q^{\bar{\delta}}}{\partial q^{\delta}} \frac{\partial q^{\alpha}}{\partial q^{\bar{\alpha}}} \frac{\partial q^{\beta}}{\partial q^{\bar{\beta}}} \cdots \tag{10}$$

Now inverting the transformation matrices ⇒

$$T^{\bar{\alpha}\bar{\beta}\cdots}_{\bar{\gamma}\bar{\delta}\cdots} = T^{\alpha\beta\cdots}_{\gamma\delta\cdots} \frac{\partial q^{\gamma}}{\partial q^{\bar{\gamma}}} \frac{\partial q^{\delta}}{\partial q^{\bar{\delta}}} \frac{\partial q^{\bar{\alpha}}}{\partial q^{\alpha}} \frac{\partial q^{\bar{\beta}}}{\partial q^{\beta}} \cdots \tag{11}$$

Note that in equation (11), the order that the transformations appear doesn't make a difference because they are independent of each other and connected through Einstein's summation convention. Also, note that the transformation matrices are not matrix multiplied together because the indices are all different.

Equation (12) shows the invariance of a second order tensor with covariant components. When the basis transforms, the coefficients transform in a way that keeps the tensor invariant.

$$\boldsymbol{T} = T^{ij}\boldsymbol{e_i} \otimes \boldsymbol{e_j} = T^{\bar{\imath}\bar{\jmath}}\boldsymbol{e_{\bar{\imath}}} \otimes \boldsymbol{e_{\bar{\jmath}}} \tag{12}$$

Matrix Representation of Second Order Tensors

Equation (13) shows a 2nd order contravariant tensor and how it converts into a matrix equation – matrix form. Let i and j be free indices ⇒

$$T^{kl}\boldsymbol{e_k} \otimes \boldsymbol{e_l} = \sum_{kl} T^{kl} \begin{pmatrix} k^{th}\ basis\ vector \\ as\ column\ vector \end{pmatrix} \begin{pmatrix} l^{th}\ basis\ vector \\ as\ row\ vector \end{pmatrix} = T^{kl}\boldsymbol{e_k^T}\boldsymbol{e_l} = T^{kl}E_{ik}^T E_{lj} \tag{13}$$

Now rearrange the components of equation (13) to get the correct order to define matrix multiplies ⇒

$$\boldsymbol{T} = T^{kl}\boldsymbol{e_k^T}\boldsymbol{e_l} = E_{ik}^T T^{kl} E_{lj} = E^T T E \tag{14}$$

Second Order Tensor Configurations

Generalizing equation (14) ⇒

$$\boldsymbol{T} = T(i,j)\boldsymbol{f}(\boldsymbol{i}) \otimes \boldsymbol{h}(\boldsymbol{j}) = F^T T(i,j) H \tag{15}$$

where

$T(i,j)$ gives a general tensor where the various indices can be lower or upper indices.
$\boldsymbol{f}(\boldsymbol{i})$ and $\boldsymbol{h}(\boldsymbol{j})$ are basis vectors and can be basis or reciprocal basis vectors.

Using Einstein's summation convention,

$\begin{Bmatrix} \boldsymbol{e_p} \\ \boldsymbol{\omega^p} \end{Bmatrix}$ basis vectors sum with $\begin{Bmatrix} up\ p\ index\ of\ T \\ down\ p\ index\ of\ T \end{Bmatrix}$

$\boldsymbol{e_p}$ is the p^{th} row of E
$\boldsymbol{\omega^i}$ is the p^{th} row of W

$F = \begin{Bmatrix} E \\ W \end{Bmatrix}$

$H = \begin{Bmatrix} E \\ W \end{Bmatrix}$

Table 1 shows the combinations of $T(i,j)$ in terms of their $\begin{Bmatrix} up \\ down \end{Bmatrix}$ components.

$T(i,j)$	j	u	d
i			
u		$T(u,u) = T^{ij}$	$T(u,d) = T_j^i$
d		$T(d,u) = T_i^j$	$T(d,d) = T_{ij}$

Table 1

Table 2 shows the possible configurations of equation (15).

$T(i,j)$	H	E	W
F			
E		$T(u,u) = T^{ij}$	$T(u,d) = T_j^i$
W		$T(d,u) = T_i^j$	$T(d,d) = T_{ij}$

Table 2

The algorithm to compute $F^T T(i,j) H$ is given by equation (16).

$$\boldsymbol{T} = F_{ik}^T T_{kl} H_{lj} = F_{ki} T_{kl} H_{lj} \tag{16}$$

Transformation of Second Order Tensors

Tensors are invariant and independent of a given coordinate system ⇒

$$\boldsymbol{T} = F^T T(i,j) H = \bar{F}^T T(\bar{\imath},\bar{\jmath}) \bar{H} \tag{17}$$

To transform the coordinate system⇒

$$\bar{F} = M_1 F \tag{18}$$

where

$$M_1 = \begin{Bmatrix} A \\ B^T \end{Bmatrix}$$

$$\bar{H} = M_2 H \tag{19}$$

where

$$M_2 = \begin{Bmatrix} A \\ B^T \end{Bmatrix}$$

Using equations (18) and (19) in equation (17) ⇒

$$\boldsymbol{T} = F^T T(i,j) H = \bar{F}^T T(\bar{\imath},\bar{\jmath}) \bar{H} = (M_1 F)^T T(\bar{\imath},\bar{\jmath}) M_2 H$$

$$= F^T M_1^T T(\bar{\imath},\bar{\jmath}) M_2 H \tag{20}$$

Equation (20) ⇒

$$T(i,j) = M_1^T \bar{T}(i,j) M_2 \tag{21}$$

Solving for $T(\bar{\imath},\bar{\jmath})$ ⇒

$$T(\bar{\imath},\bar{\jmath}) = [M_1^T]^{-1} T(i,j) [M_2]^{-1} = ([M_1]^{-1})^T T(i,j) [M_2]^{-1} \tag{22}$$

Table 3 shows the possible configurations of equation (22).

$T(i,j)$	$[M_2]^{-1}$	$A^{-1} = B$	$[B^T]^{-1} = A^T$
$[M_1]^{-1}$			
$[A]^{-1} = B$		$T(u,u) = T^{ij}$	$T(u,d) = T_j^i$
$[B^{-1}]^T = A^T$		$T(d,u) = T_i^j$	$T(d,d) = T_{ij}$

Table 3

Equation (23) shows an algorithm similar to equation (16) for the transformation of second order tensor components.

$$[([M_1]^{-1})^T T(i,j) [M_2]^{-1}]_{ij} = ([M_1]^{-1})_{ik}^T T_{kl} ([M_2]^{-1})_{lj} = M_{ki}^{-1} T_{kl} M_{lj}^{-1} \tag{23}$$

We now go over the various possibilities in Table 3.

Contravariant Components T^{ij}

$$T = T(u,u) = T^{ij}$$

From Table 3 ⇒

$[M_1]^{-1} = B$
$[M_2]^{-1} = B$

Equation (23) ⇒

$$\bar{T} = B^T T B \tag{24}$$

The metric with the same configuration as $T(u,u) \equiv g^{ij} = G^{-1}$ in equation (24) ⇒

$\bar{G}^{-1} = B^T G^{-1} B$

(25)

Equation (25) is the same equation that we have obtained previously as shown in Figure 1 below – see red arrow from node 4 to node 3. Figure 1 shows the relationships between coordinate transformations, basis vectors, reciprocal basis vectors - one forms -, and the metric.

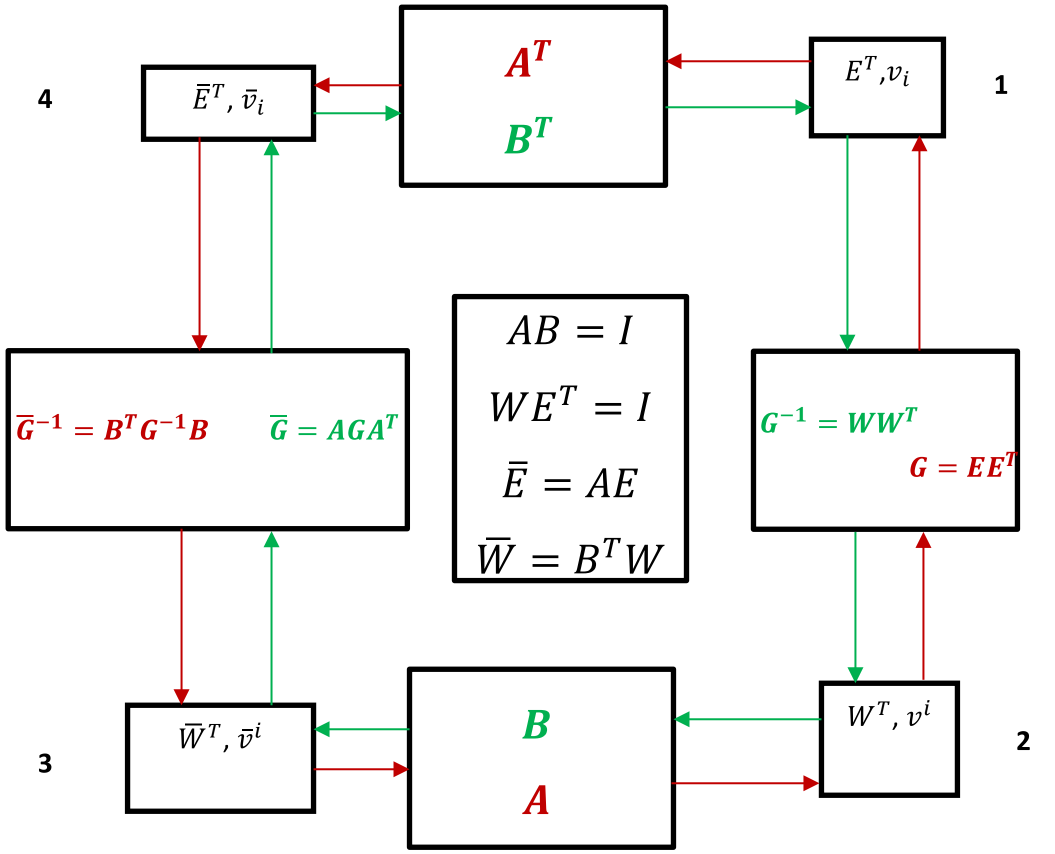

Figure 1

Covariant Components T_{ij}

$T = T(d, d)$

From Table 3 $\Rightarrow$

$[M_1]^{-1} = A^T$
$[M_2]^{-1} = A^T$

Equation (23) $\Rightarrow$

$\bar{T} = ATA^T$

(26)

The metric corresponding to $T(d,d) \equiv g_{ij} = G \;\Rightarrow$

$$\bar{G} = AGA^T \tag{27}$$

Equation (27) is the same equation previously obtained – see Figure 1 green arrow - node 3 to node 4.

Mixed Components T^i_j

$T = T(u,d)$

From Table 3 ⇒

$[M_1]^{-1} = B$
$[M_2]^{-1} = A^T$

Equation (23) ⇒

$$\bar{T} = B^T T A^T \tag{28}$$

Mixed Components T_i^j

$T = T(d,u)$

From Table 3 ⇒

$[M_1]^{-1} = A^T$
$[M_2]^{-1} = B$

Equation (23) ⇒

$$\bar{T} = ATB \tag{29}$$

Index Raising and Lowering

This section will map out how to transform T^{ij} into the other three configurations and visa-versa. First map out how to lower the first index of T^{ij} ⇒

From Table 2, the bases for $T^{ij} \;\Rightarrow E^T T^{ij} E$
From Table 2, the bases for $T_i^j \;\Rightarrow W^T T_i^j E \;\Rightarrow$

Invariance of the tensor ⇒

$E^T T^{ij} E = W^T T_i^j E \;\Rightarrow$

$E^T T^{ij} = W^T T_i^j \ \Rightarrow$

But $EW^T = I \ \Rightarrow$

$EE^T T^{ij} = T_i^j = GT^{ij}$

(30)

where

$G = EE^T \ \equiv$ the metric

Equation (30) lowers the first index of T^{ij}

Equation (30) can be inverted ⇒

$T^{ij} = G^{-1} T_i^j$

(31)

where

$G^{-1} = WW^T \ \equiv$ the inverse metric

Equation (31) raises the first index of T_i^j

To lower the second index of $T^{ij} \ \Rightarrow$

From Table 2, the bases for $T_j^i \ \Rightarrow E^T T_i^j W \ \Rightarrow$

$E^T T^{ij} E = E^T T_j^i W \ \Rightarrow$

$T^{ij} E = T_j^i W$

But $WE^T = I \ \Rightarrow$

$T^{ij} EE^T = T_j^i \ \Rightarrow$

$T_j^i = T^{ij} G$

(32)

Equation (32) lowers the second index of T^{ij}

Equation (32) can be inverted ⇒

$T^{ij} = T_j^i G^{-1}$

(33)

Equation (33) raises the second index of T_j^i

To lower both indices of T^{ij} $\Rightarrow$

From Table 2, the bases for T_{ij} $\Rightarrow$ $W^T T_{ij} W$

$E^T T^{ij} E = W^T T_{ij} W$ $\Rightarrow$

$E E^T T^{ij} E = T_{ij} W$ $\Rightarrow$

$$E E^T T^{ij} E E^T = T_{ij} = G T^{ij} G \tag{34}$$

Equation (34) lowers both indices of T^{ij}

Equations (34) can be inverted $\Rightarrow$

$$T^{ij} = G^{-1} T_{ij} G^{-1} \tag{35}$$

Equation (35) raises both indices of T_{ij}

Summary Raising and Lowering Operations

Table 4 gives a summary of raising and lowering operations for second order tensors.

$T_i^j = G T^{ij}$	$T^{ij} = G^{-1} T_i^j$
$T_j^i = T^{ij} G$	$T^{ij} = T_j^i G^{-1}$
$T_{ij} = G T^{ij} G$	$T^{ij} = G^{-1} T_{ij} G^{-1}$

Table 4

Contravariant Components Example

The basis set for polar coordinates are shown in equation (36).

$$E(r,\theta) = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -r\ sin(\theta) & r\ cos(\theta) \end{bmatrix} \tag{36}$$

For

$r = 2$

$\theta = \dfrac{\pi}{3}$

$$E\left(2,\frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$\boldsymbol{e_1} = [0.500000 \quad 0.866025]$$

$$\boldsymbol{e_2} = [-1.732051 \quad 1.000000]$$

$$T^{ij} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{37}$$

Compute $\boldsymbol{T}$ using the tensor product.

$$\boldsymbol{T} = T^{ij}\boldsymbol{e_i} \otimes \boldsymbol{e_j} = T^{ij}\boldsymbol{e_i^T}\boldsymbol{e_j}$$

$$= (1)\boldsymbol{e_1^T}\boldsymbol{e_1} + (2)\boldsymbol{e_1^T}\boldsymbol{e_2} + (3)\boldsymbol{e_2^T}\boldsymbol{e_1} + (4)\boldsymbol{e_2^T}\boldsymbol{e_2}$$

$$= (1)\begin{bmatrix} 0.500000 \\ 0.866025 \end{bmatrix}[0.500000 \quad 0.866025] + (2)\begin{bmatrix} 0.500000 \\ 0.866025 \end{bmatrix}[-1.732051 \quad 1.000000]$$

$$+ (3)\begin{bmatrix} -1.732051 \\ 1.000000 \end{bmatrix}[0.500000 \quad 0.866025] + (4)\begin{bmatrix} -1.732051 \\ 1.000000 \end{bmatrix}[-1.732051 \quad 1.000000]$$

$$= (1)\begin{bmatrix} 0.250000 & 0.433013 \\ 0.433013 & 0.750000 \end{bmatrix} + (2)\begin{bmatrix} -0.866025 & 0.500000 \\ -1.500000 & 0.866025 \end{bmatrix} + (3)\begin{bmatrix} -0.866025 & -1.500000 \\ 0.500000 & 0.866025 \end{bmatrix}$$

$$+(4)\begin{bmatrix} 3.000000 & -1.732051 \\ -1.732051 & 1.000000 \end{bmatrix} = \begin{bmatrix} 7.919873 & -9.995191 \\ -7.995191 & 9.080127 \end{bmatrix} \tag{38}$$

Now perform the same calculation using equation (16) ⇒

$$\boldsymbol{T} = E^T T E = \begin{bmatrix} 0.500000 & -1.732051 \\ 0.866025 & 1.000000 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$= \begin{bmatrix} 7.919873 & -9.995191 \\ -7.995191 & 9.080127 \end{bmatrix} \tag{39}$$

Equation (39) is the same as equation (38) as expected.

Transformation of Contravariant Components

Equation (34) shows the coordinate transform equations for a different coordinate system.

$$r = \sqrt{\bar{r}}$$
$$\theta = \sqrt{\bar{\theta}}$$

$\bar{r} = r^2$

$\bar{\theta} = \theta^2$

(40)

Using the same r and θ from the previous calculation ⇒

$\bar{r} = 4$

$\bar{\theta} = \frac{\pi^2}{9}$

(41)

Equation (42) shows the A matrix to transform from polar coordinates this coordinate system.

$$A = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix} \tag{42}$$

Equation (43) shows the basis transformation equation.

$$\bar{E} = AE = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} = \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix} \tag{43}$$

$\boldsymbol{e_1} = [0.125000 \quad 0.216506]$

$\boldsymbol{e_2} = [-0.826993 \quad 0.477465]$

From Table 3, $T(u,u) \rightarrow \bar{T} = B^T TB$ ⇒

$$B = \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix} \tag{44}$$

$$\bar{T} = B^T TB = \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix} = \begin{bmatrix} 16.000000 & 16.755161 \\ 25.132741 & 17.545963 \end{bmatrix} \tag{45}$$

$\boldsymbol{T} = T^{\bar{\imath}\bar{\jmath}} \boldsymbol{e_{\bar{\imath}}} \otimes \boldsymbol{e_{\bar{\jmath}}} = T^{\bar{\imath}\bar{\jmath}} \boldsymbol{e_{\bar{\imath}}^T} \boldsymbol{e_{\bar{\jmath}}}$

$= (16)\boldsymbol{e_{\bar{1}}^T e_{\bar{1}}} + (16.755161)\boldsymbol{e_{\bar{1}}^T e_{\bar{2}}} + (25.132741)\boldsymbol{e_{\bar{2}}^T e_{\bar{1}}} + (17.545963)\boldsymbol{e_{\bar{2}}^T e_{\bar{2}}}$

$= (16) \begin{bmatrix} 0.125000 \\ 0.216506 \end{bmatrix} [0.125000 \quad 0.216506] + (16.755161) \begin{bmatrix} 0.125000 \\ 0.216506 \end{bmatrix} [-0.826993 \quad 0.477465]$

$$+(25.132741)\begin{bmatrix}-0.826993\\0.477465\end{bmatrix}[0.125000 \quad 0.216506]$$

$$+(17.545963)\begin{bmatrix}-0.826993\\0.477465\end{bmatrix}[-0.826993 \quad 0.477465] =$$

$$(16)\begin{bmatrix}0.015625 & 0.027063\\0.027063 & 0.046875\end{bmatrix} + (16.755161)\begin{bmatrix}-0.103374 & 0.059683\\-0.179049 & 0.103374\end{bmatrix}$$

$$+(25.132741)\begin{bmatrix}-0.103374 & -0.179049\\0.059683 & 0.103374\end{bmatrix} + (17.545963)\begin{bmatrix}0.683918 & -0.394860\\-0.394860 & 0.227973\end{bmatrix}$$

$$= \begin{bmatrix}7.919873 & -9.995191\\-7.995191 & 9.080127\end{bmatrix} \tag{46}$$

Equation (46) is the same as equation (39) as expected. Computing using the matrix approach ⇒

$$\boldsymbol{T} = \bar{E}^T\bar{T}\bar{E} = \begin{bmatrix}0.125000 & -0.826993\\0.216506 & 0.477465\end{bmatrix}\begin{bmatrix}16.000000 & 16.755161\\25.132741 & 17.545963\end{bmatrix}\begin{bmatrix}0.125000 & 0.216506\\-0.826993 & 0.477465\end{bmatrix}$$

$$\begin{bmatrix}7.919873 & -9.995191\\-7.995191 & 9.080127\end{bmatrix} \tag{47}$$

Equation (47) is the same as equation (46) as expected. For all configurations test drivers in both unprimed and primed system see [2]

Third Order Tensors

Second order tensors were covered in the last section. Going back to the definition of the tensor product; $Object`_1 \otimes Object`_2$ is computed by multiplying each element of $Object`_1$ by $Object`_2$. In the second order case it is.

$$\boldsymbol{e_i} \otimes \boldsymbol{e_j} = [\boldsymbol{e_i}]^{\boldsymbol{T}}\boldsymbol{e_j} = E_{il}E_{jm} \tag{48}$$

where

$l, m = 1, n \equiv$ basis vector length

A third order basis is a tensor product of three basis vectors as shown in equation (49).

$$\boldsymbol{e_i} \otimes \boldsymbol{e_j} \otimes \boldsymbol{e_k} = E_{il}E_{jm}E_{kp} \tag{49}$$

Equation (50) is the general equation for a third order tensor.

[2] Tensor Transform Code

$$\boldsymbol{T} = T(i,j,k)\boldsymbol{l}(i) \otimes \boldsymbol{f}(j) \otimes \boldsymbol{h}(k) = T(i,j,k)L_{il}F_{jo}H_{kp} = T(i,j,k)L(i,l)F(j,o)H(k,p) \tag{50}$$

where

$T(i,j,k)$ are the tensor coordinates

$\boldsymbol{l}(i), \boldsymbol{f}(j), \boldsymbol{h}(k)$ are basis or reciprocal basis vectors

$L_{il} = L(i,l) \equiv\ i^{th}$ row of $L = \boldsymbol{l}(i)$

$F_{jo} = F(j,o) \equiv\ j^{th}$ row of $F =\ \boldsymbol{f}(j)$

$H_{kp} = H(k,p) \equiv\ k^{th}$ row of $H = \boldsymbol{h}(k)$

$l, o, p = 1, \cdots, n$

$$\begin{cases} L = E, i\ in\ up\ position\ in\ T, else\ L = W \\ F = E,\ j\ in\ up\ position\ in\ T, else\ F = W \\ H = E,\ k\ in\ up\ position\ in\ T, else\ H = W \end{cases}$$

Equation (50) ⇒

$$\boldsymbol{T} = T(i,j,k)\boldsymbol{l}(i) \otimes \boldsymbol{f}(j) \otimes \boldsymbol{h}(k) = [T(i,j,k)L(i,l)]F(j,o)H(k,p)$$

$$= [T(i,j,k) \cdot L(i,l)]F(j,o)H(k,p) \tag{51}$$

$T(i,j,k) \cdot L(i,l)$ is a block vector inner product – it behaves like a normal inner product but $T(i,j,k)$ is a block vector, so each element $T(i,j,k)$ is a matrix for a fixed i ⇒

$$T(i,j,k) \cdot L = [T(1,j,k) \quad \cdots \quad T(n,j,k)] \cdot \begin{bmatrix} L_{11} & \cdots & L_{1l} \\ \vdots & \ddots & \vdots \\ L_{n1} & \cdots & L_{nl} \end{bmatrix} \Rightarrow$$

For column p ⇒

$$T_p(j,k) = T(1,j,k)L_{1p} + \cdots + T(n,j,k)L_{np} = [T(1,j,k) \quad \cdots \quad T(n,j,k)] \cdot col(L,p) \tag{52}$$

Using equation (52), equation (51) ⇒

$$\boldsymbol{T} = T(i,j,k)\boldsymbol{l}(i) \otimes \boldsymbol{f}(j) \otimes \boldsymbol{h}(k) = [T_1(j,k) \quad \cdots \quad T_n(j,k)]F_{jo}H_{kp}$$

$$= [F^T T_1(j,k)H \quad \cdots \quad F^T T_n(j,k)H] = [F^T T_1 H \quad \cdots \quad F^T T_n H] \Rightarrow$$

$$[F^T] \otimes [T_1(j,k) \quad \cdots \quad T_n(j,k)] \otimes [H] = [F^T] \otimes [T(i,j,k) \cdot L] \otimes [H] \tag{52}$$

Note: $[F^T]$ and $[H]$ are 1×1 matrices containing the F^T and H matrices, so the tensor product distributes the F^T and H matrices over the $T_p(j,k)$ matrices. The $' \cdot '$ specifies a block matrix inner product and the $F^T T_n H$ components are regular matrix products.

$$M \otimes [N_1 \quad N_2] \neq [M] \otimes [N_1 \quad N_2]$$

$$M \otimes [N_1 \quad N_2] = M_{ij}[N_1 \quad N_2] = \begin{bmatrix} M_{11}[N_1 \quad N_2] & \cdots & M_{1n}[N_1 \quad N_2] \\ \vdots & \ddots & \vdots \\ M_{n1}[N_1 \quad N_2] & \cdots & M_{nn}[N_1 \quad N_2] \end{bmatrix} \equiv \text{block matrix}$$

$[M] \otimes [N_1 \quad N_2] = [MN_1 \quad MN_2] \equiv$ each element of $[M]$, i.e. M, is multiplied by each element of $[N_1 \quad N_2]$

Transformation of Third Order Tensors

Equate the equation (52) between the unprimed and primed coordinate systems ⇒

$$\boldsymbol{T} = [F^T] \otimes [T(i,j,k) \cdot L] \otimes [H] = [\bar{F}^T] \otimes \left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right) \cdot \bar{L}\right] \otimes [\bar{H}] \tag{53}$$

Now

$$\bar{F} = M_1 F \tag{54}$$

$$\bar{H} = M_2 H \tag{55}$$

$$\bar{L} = M_3 L \tag{56}$$

where

$$M_1, M_2, M_3 = \begin{cases} A \; for \; basis \; E \\ B^T \; for \; reciprocal \; basis \; W \end{cases}$$

Equations (53) – (56) ⇒

$$[F^T] \otimes [T(i,j,k) \cdot L] \otimes [H] = [M_1 F]^T \otimes \left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right) \cdot \bar{L}\right] \otimes [M_2 H] \;\Rightarrow$$

$$[F^T] \otimes [T(i,j,k) \cdot L] \otimes [H] = [F^T] \cdot [M_1^T] \otimes \left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right) \cdot \bar{L}\right] \otimes [M_2] \cdot [H] \;\Rightarrow$$

$$[T(i,j,k) \cdot L] = [M_1^T] \otimes \left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right) \cdot \bar{L}\right] \otimes [M_2] \tag{57}$$

Solving for $\left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right)\cdot\bar{L}\right]$ in equation (57) ⇒

$$[I]\otimes\left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right)\cdot\bar{L}\right]\otimes[I]=\left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right)\cdot\bar{L}\right]=[M_1^T]^{-1}\otimes\ [T(i,j,k)\cdot L]\otimes[M_2]^{-1} \tag{58}$$

Note:

$[I]\otimes[M]$ ≡ multiply each element of $[I]$ with each element of $[M]$ ⇒

$[I]\otimes[M]=[I\cdot M]=[M]$

Solving for $T\left(\bar{\imath},\bar{\jmath},\bar{k}\right)$ in equation (58) ⇒

$$T\left(\bar{\imath},\bar{\jmath},\bar{k}\right)=\left[T\left(\bar{\imath},\bar{\jmath},\bar{k}\right)\cdot\bar{L}\right]\cdot[\bar{L}^{-1}] \tag{59}$$

$T\left(\bar{\imath},\bar{\jmath},\bar{k}\right)$ is computed using equations (58) and (59).

Third Order Tensor Example

Convert cartesian coordinates to polar coordinates, then start with polar coordinates.

$$A=\begin{bmatrix}cos(\theta) & sin(\theta)\\ -r\ sin(\theta) & r\ cos(\theta)\end{bmatrix}$$

$$B=\begin{bmatrix}cos(\theta) & -\frac{sin(\theta)}{r}\\ sin(\theta) & \frac{cos(\theta)}{r}\end{bmatrix} \tag{60}$$

$$E=AI \tag{61}$$

$$W=B^TI=\begin{bmatrix}cos(\theta) & sin(\theta)\\ -\frac{sin(\theta)}{r} & \frac{cos(\theta)}{r}\end{bmatrix} \tag{62}$$

Let

$r=2$

$\theta=\frac{\pi}{3}$

$$E=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\sqrt{3} & 1\end{bmatrix}=\begin{bmatrix}0.500000 & 0.866025\\ -1.732051 & 1.000000\end{bmatrix}$$

(63)

$$\boldsymbol{e_1} = [0.500000 \quad 0.866025]$$

$$\boldsymbol{e_2} = [-1.732051 \quad 1.000000]$$

(64)

$$W = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix}$$

$$\boldsymbol{\omega^1} = [0.500000 \quad 0.866025]$$

$$\boldsymbol{\omega^2} = [-0.433013 \quad 0.250000]$$

(65)

Specify a third order tensor in polar coordinates.

$$T = \left[\begin{bmatrix} T_1^{11} & T_1^{12} \\ T_2^{11} & T_2^{12} \end{bmatrix} \quad \begin{bmatrix} T_1^{21} & T_1^{22} \\ T_2^{21} & T_2^{22} \end{bmatrix}\right] = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\right]$$

(66)

Form basis terms using tensor products and add

For a detailed calculation see - [Appendix A]

$$\boldsymbol{T} = T_j^{ik}\boldsymbol{e_i} \otimes \boldsymbol{\omega^j} \otimes \boldsymbol{e_k} = T_1^{11}\boldsymbol{e_1} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_1} + T_1^{12}\boldsymbol{e_1} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_2} + T_2^{11}\boldsymbol{e_1} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_1}$$

$$+T_2^{12}\boldsymbol{e_1} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_2} + T_1^{21}\boldsymbol{e_2} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_1} + T_2^{21}\boldsymbol{e_2} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_2} + T_2^{21}\boldsymbol{e_2} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_1}$$

$$+T_2^{22}\boldsymbol{e_2} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_2} = \left[\begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} \quad \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix}\right]$$

(67)

Now we show how to compute a third order basis term.

$$\boldsymbol{e_1} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_2} = \boldsymbol{e_1} \otimes [\boldsymbol{\omega^1} \otimes \boldsymbol{e_2}] = \boldsymbol{e_1} \otimes \left[[\boldsymbol{\omega^1}]^{\boldsymbol{T}}\boldsymbol{e_2}\right] = \left[E_{11}\left[[\boldsymbol{\omega^1}]^{\boldsymbol{T}}\boldsymbol{e_2}\right] \quad E_{12}\left[[\boldsymbol{\omega^1}]^{\boldsymbol{T}}\boldsymbol{e_2}\right]\right]$$

$$= \left[0.500000 \begin{bmatrix} 0.500000 \\ 0.866025 \end{bmatrix} [-1.732051 \quad 1.000000] \quad 0.866025 \begin{bmatrix} 0.500000 \\ 0.866025 \end{bmatrix} [-1.732051 \quad 1.000000]\right]$$

$$= \left[0.500000 \begin{bmatrix} -0.866025 & 0.500000 \\ -1.500000 & 0.866025 \end{bmatrix} \quad 0.866025 \begin{bmatrix} -0.866025 & 0.500000 \\ -1.500000 & 0.866025 \end{bmatrix}\right]$$

$$= \left[\begin{bmatrix} -0.433013 & 0.250000 \\ -0.750000 & 0.433013 \end{bmatrix} \quad \begin{bmatrix} -0.750000 & 0.433013 \\ -1.299038 & 0.750000 \end{bmatrix}\right]$$

(68)

Notice that equation the $\boldsymbol{e_1} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_2}$ term in (68) differs from the same term in the appendix equation (A.1)

$$\left[\begin{bmatrix} -0.433013 & -0.750000 \\ -0.750000 & -1.299038 \end{bmatrix} \quad \begin{bmatrix} 0.250000 & 0.433013 \\ 0.433013 & 0.750000 \end{bmatrix}\right] \tag{A.1}$$

The terms between equation (68) and (A.1) are the same but in different positions. Because the entire expression is a third order basis, the two are equivalent just with the terms in different places. To show why this is consider equation (68).

$$\boldsymbol{\omega^1} \otimes \boldsymbol{e_2} = [\boldsymbol{\omega^1} \otimes \boldsymbol{e_2}] = [\boldsymbol{\omega^1}]^T \boldsymbol{e_2} = \begin{bmatrix} W_{11} \\ W_{12} \end{bmatrix} [E_{21} \quad E_{22}] = \begin{bmatrix} W_{11}E_{21} & W_{11}E_{22} \\ W_{12}E_{21} & W_{12}E_{22} \end{bmatrix}$$

$$E_{11}[\boldsymbol{\omega^1} \otimes \boldsymbol{e_2}] = \begin{bmatrix} E_{11}W_{11}E_{21} & E_{11}W_{11}E_{22} \\ E_{11}W_{12}E_{21} & E_{11}W_{12}E_{22} \end{bmatrix}$$

$$E_{12}[\boldsymbol{\omega^1} \otimes \boldsymbol{e_2}] = E_{12} \begin{bmatrix} W_{11}E_{21} & W_{11}E_{22} \\ W_{12}E_{21} & W_{12}E_{22} \end{bmatrix} = \begin{bmatrix} E_{12}W_{11}E_{21} & E_{12}W_{11}E_{22} \\ E_{12}W_{12}E_{21} & E_{12}W_{12}E_{22} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{e_1} \otimes [\boldsymbol{\omega^1} \otimes \boldsymbol{e_2}] = \left[\begin{bmatrix} E_{11}W_{11}E_{21} & E_{11}W_{11}E_{22} \\ E_{11}W_{12}E_{21} & E_{11}W_{12}E_{22} \end{bmatrix} \quad \begin{bmatrix} E_{12}W_{11}E_{21} & E_{12}W_{11}E_{22} \\ E_{12}W_{12}E_{21} & E_{12}W_{12}E_{22} \end{bmatrix}\right] \tag{69}$$

Notice if we change the way the equation (69) is bracketed ⇒

$$[\boldsymbol{e_1} \otimes \boldsymbol{\omega^1}] \otimes \boldsymbol{e_2} \ \Rightarrow$$

$$\boldsymbol{e_1} \otimes \boldsymbol{\omega^1} = [\boldsymbol{e_1}]^T \boldsymbol{\omega^1} = \begin{bmatrix} E_{11} \\ E_{12} \end{bmatrix} [W_{11} \quad W_{12}] = \begin{bmatrix} E_{11}W_{11} & E_{11}W_{12} \\ E_{12}W_{11} & E_{12}W_{12} \end{bmatrix}$$

$$[\boldsymbol{e_1} \otimes \boldsymbol{\omega^1}]E_{21} = \begin{bmatrix} E_{11}W_{11} & E_{11}W_{12} \\ E_{12}W_{11} & E_{12}W_{12} \end{bmatrix} E_{21} = \begin{bmatrix} E_{11}W_{11}E_{21} & E_{11}W_{12}E_{21} \\ E_{12}W_{11}E_{21} & E_{12}W_{12}E_{21} \end{bmatrix}$$

$$[\boldsymbol{e_1} \otimes \boldsymbol{\omega^1}]E_{22} = \begin{bmatrix} E_{11}W_{11} & E_{11}W_{12} \\ E_{12}W_{11} & E_{12}W_{12} \end{bmatrix} E_{22} = \begin{bmatrix} E_{11}W_{11}E_{22} & E_{11}W_{12}E_{22} \\ E_{12}W_{11}E_{22} & E_{12}W_{12}E_{22} \end{bmatrix} \Rightarrow$$

$$[\boldsymbol{e_1} \otimes \boldsymbol{\omega^1}] \otimes \boldsymbol{e_2} = \left[\begin{bmatrix} E_{11}W_{11}E_{21} & E_{11}W_{12}E_{21} \\ E_{12}W_{11}E_{21} & E_{12}W_{12}E_{21} \end{bmatrix} \quad \begin{bmatrix} E_{11}W_{11}E_{22} & E_{11}W_{12}E_{22} \\ E_{12}W_{11}E_{22} & E_{12}W_{12}E_{22} \end{bmatrix}\right] \tag{70}$$

Comparing equations (69) and (70), we get the same elements but, they are in different positions as shown below.

$$\boldsymbol{e_1} \otimes [\boldsymbol{\omega^1} \otimes \boldsymbol{e_2}] = \left[\begin{bmatrix} E_{11}W_{11}E_{21} & E_{11}W_{11}E_{22} \\ E_{11}W_{12}E_{21} & E_{11}W_{12}E_{22} \end{bmatrix} \quad \begin{bmatrix} E_{12}W_{11}E_{21} & E_{12}W_{11}E_{22} \\ E_{12}W_{12}E_{21} & E_{12}W_{12}E_{22} \end{bmatrix}\right]$$

$$[\boldsymbol{e_1} \otimes \boldsymbol{\omega^1}] \otimes \boldsymbol{e_2} = \left[\begin{bmatrix} E_{11}W_{11}E_{21} & E_{11}W_{12}E_{21} \\ E_{12}W_{11}E_{21} & E_{12}W_{12}E_{21} \end{bmatrix} \quad \begin{bmatrix} E_{11}W_{11}E_{22} & E_{11}W_{12}E_{22} \\ E_{12}W_{11}E_{22} & E_{12}W_{12}E_{22} \end{bmatrix}\right]$$

The basis is the entire block vector, so either basis will do – just be aware of the difference in position of the elements and keep the position of the elements consistent. For the specific case from equations (68) and (A.1) ⇒

$$\left[\left[\begin{bmatrix}-0.433013\\-0.750000\end{bmatrix}\quad\begin{bmatrix}0.250000\\0.433013\end{bmatrix}\right]\quad\left[\begin{bmatrix}-0.750000\\-1.299038\end{bmatrix}\quad\begin{bmatrix}0.433013\\0.750000\end{bmatrix}\right]\right]\boldsymbol{e_1}\otimes[\boldsymbol{\omega^1}\otimes\boldsymbol{e_2}]$$

$$\left[\left[\begin{bmatrix}-0.433013 & -0.750000\\-0.750000 & -1.299038\end{bmatrix}\quad\begin{bmatrix}0.250000 & 0.433013\\0.433013 & 0.750000\end{bmatrix}\right]\right][\boldsymbol{e_1}\otimes\boldsymbol{\omega^1}]\otimes\boldsymbol{e_2}$$

Tensor Computation using $\boldsymbol{F^T T_n H}$

From equation (52),

$$\boldsymbol{T} = T^i_{jk}\boldsymbol{e_i}\otimes\boldsymbol{\omega^j}\otimes\boldsymbol{e_k} = F^T\otimes[T(i,j,k)\cdot L]\otimes H$$

But

$\boldsymbol{e_i} \Rightarrow L = E$
$\boldsymbol{\omega^j} \Rightarrow F = W$
$\boldsymbol{e_k} \Rightarrow H = E$

So,

$$\boldsymbol{T} = T^i_{jk}\boldsymbol{e_i}\otimes\boldsymbol{\omega^j}\otimes\boldsymbol{e_k} = W^T\otimes[T(i,j,k)\cdot E]\otimes E$$

$$= [W^T T(i,j,k)\cdot col(E,1)E \quad W^T T(i,j,k)\cdot col(E,2)E] = [W^T T_1 E \quad W^T T_2 E] \tag{71}$$

$$T_1 = [T\cdot E]_1 = T\cdot col(E,1) = T^{ijk}E_{i1} = T^{1jk}E_{11} + T^{2jk}E_{21}$$

$$= E_{11}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + E_{21}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = (0.500000)\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + (-1.732051)\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}$$

$$= \begin{bmatrix}-8.160254 & -9.392305\\-10.624356 & -11.856406\end{bmatrix} \tag{72}$$

$$T_2 = [T\cdot E]_2 = T\cdot col(E,2) = T^{ijk}E_{i2} = T^{1jk}E_{12} + T^{2jk}E_{22}$$

$$= E_{12}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + E_{22}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = 0.866025\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + (1.000000)\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = \begin{bmatrix}5.866025 & 7.732051\\9.598076 & 11.464102\end{bmatrix} \tag{73}$$

$$T^i_{jk}\boldsymbol{e_i}\otimes\boldsymbol{\omega^j}\otimes\boldsymbol{e_k} = [W^T T_1 E \quad W^T T_2 E] \;\Rightarrow$$

$$W^T T_1 E = \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix} \begin{bmatrix} -8.160254 & -9.392305 \\ -10.624356 & -11.856406 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} \tag{74}$$

$$= \begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} \tag{75}$$

$$W^T T_2 E = \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix} \begin{bmatrix} 5.866025 & 7.732051 \\ 9.598076 & 11.464102 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$= \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix} \tag{76}$$

Putting equations (75) and (76) into equation (74) ⇒

$$T^i_{jk} \boldsymbol{e_i} \otimes \boldsymbol{\omega^j} \otimes \boldsymbol{e_k} = \left[\begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} \quad \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix} \right] \tag{77}$$

Equation (77) is the same as equation (67) as expected.

Transformation Example

We will use the tensor components from the previous example, but in a different configuration – different up/down pattern of the indices as shown in equation (78).

$$T = T^i_{jk} = T(u, d, u) = \left[\begin{bmatrix} T_1^{11} & T_1^{12} \\ T_2^{11} & T_2^{12} \end{bmatrix} \quad \begin{bmatrix} T_1^{21} & T_1^{22} \\ T_2^{21} & T_2^{22} \end{bmatrix} \right] = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \right] \tag{78}$$

From the earlier 2nd order tensor transform example.

$$\bar{r} = 4$$

$$\bar{\theta} = \frac{\pi^2}{9}$$

$$E = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$W = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix}$$

$$B = \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix}$$

Because the coordinate components are of the form $T^{i}_{jk} = T(u, d, u)$; Table 3 ⇒

$$[M_1]^{-1} = A^T \;\; \Rightarrow$$

$$[M_1^{-1}]^T = A$$

$$[M_2]^{-1} = B$$

$$L = E \tag{79}$$

Putting equations (79) into equations (58) and (59)⇒

$$\left[T\left(\bar{\imath}, \bar{\jmath}, \bar{k}\right) \cdot \bar{E}\right] = A \otimes \; [T(i, j, k) \cdot E] \otimes B \tag{80}$$

$$T\left(\bar{\imath}, \bar{\jmath}, \bar{k}\right) = \left[T\left(\bar{\imath}, \bar{\jmath}, \bar{k}\right) \cdot \bar{L}\right] \cdot [\bar{L}^{-1}] = \left[T\left(\bar{\imath}, \bar{\jmath}, \bar{k}\right) \cdot \bar{E}\right] \cdot \bar{W}^T \tag{81}$$

From the previous example,

$$[T(i, j, k) \cdot E] = [T_1 \quad T_2] = \left[\begin{bmatrix} -8.160254 & -9.392305 \\ -10.624356 & -11.856406 \end{bmatrix} \quad \begin{bmatrix} 5.866025 & 7.732051 \\ 9.598076 & 11.464102 \end{bmatrix}\right] \tag{82}$$

$$\left[T\left(\bar{\imath}, \bar{\jmath}, \bar{k}\right) \cdot \bar{E}\right] = [\bar{T}_1 \quad \bar{T}_2] = [AT_1B \quad AT_2B] \tag{83}$$

$$\bar{T}_1 = AT_1B = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix} \begin{bmatrix} -8.160254 & -9.392305 \\ -10.624356 & -11.856406 \end{bmatrix} \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix}$$

$$= \begin{bmatrix} -8.160254 & -4.917799 \\ -20.291025 & -11.856406 \end{bmatrix} \tag{84}$$

$$\bar{T}_2 = AT_2B = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix} \begin{bmatrix} 5.866025 & 7.732051 \\ 9.598076 & 11.464102 \end{bmatrix} \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix}$$

$$= \begin{bmatrix} 5.866025 & 4.048492 \\ 18.330975 & 11.464102 \end{bmatrix}$$

(85)

$$[\bar{T}_1 \quad \bar{T}_2] = \left[\begin{bmatrix} -8.160254 & -4.917799 \\ -20.291025 & -11.856406 \end{bmatrix} \quad \begin{bmatrix} 5.866025 & 4.048492 \\ 18.330975 & 11.464102 \end{bmatrix}\right]$$

(86)

$$\bar{E} = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} = \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix}$$

(87)

$$\bar{W} = B^T W = \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix}$$

$$= \begin{bmatrix} 2.000000 & 3.464102 \\ -0.906900 & 0.523599 \end{bmatrix}$$

(88)

$$\boldsymbol{T} = T_{\bar{j}}^{\bar{\imath}\bar{k}} \boldsymbol{e}_{\bar{\imath}} \otimes \boldsymbol{\omega}^{\bar{j}} \otimes \boldsymbol{e}_{\bar{k}} = [\bar{W}^T \bar{T}_1 \bar{E} \quad \bar{W}^T \bar{T}_2 \bar{E}]$$

(89)

$$\bar{W}^T \bar{T}_1 \bar{E} = \begin{bmatrix} 2.000000 & 3.464102 \\ -0.906900 & 0.523599 \end{bmatrix} \begin{bmatrix} -8.160254 & -4.917799 \\ -20.291025 & -11.856406 \end{bmatrix} \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix}$$

$$= \begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix}$$

(90)

$$\bar{W}^T \bar{T}_2 \bar{E} = \begin{bmatrix} 2.000000 & 3.464102 \\ -0.906900 & 0.523599 \end{bmatrix} \begin{bmatrix} 5.866025 & 4.048492 \\ 18.330975 & 11.464102 \end{bmatrix} \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix}$$

$$= \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix}$$

(91)

$$\boldsymbol{T} = [\bar{W}^T \bar{T}_1 \bar{E} \quad \bar{W}^T \bar{T}_2 \bar{E}] = \left[\begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} \quad \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix}\right]$$

(92)

Equation (92) is the same as equation (77) as expected.

Now, use equation (59) to solve for $T(\bar{\imath}, \bar{j}, \bar{k}) \Rightarrow$

$$T_{\bar{j}}^{\bar{\imath}\bar{k}} = \left[T_{\bar{j}}^{\bar{\imath}\bar{k}} \cdot \bar{E}\right] \cdot \bar{W}^T = [\bar{T}_1 \quad \bar{T}_2] \cdot \bar{W}^T = \begin{bmatrix} 2.000000 & 3.464102 \\ -0.906900 & 0.523599 \end{bmatrix} = \Rightarrow$$

$$T_{\bar{j}}^{\bar{1}\bar{k}} = \begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} (2.000000) - \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix} (0.906900)$$

$$= \begin{bmatrix} 4.000000 & 4.188790 \\ 22.918312 & 16.000000 \end{bmatrix}$$

(93)

$$T_{\bar{j}}^{\bar{2}\bar{k}} = \begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} (3.464102) + \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix} (0.523599)$$

$$= \begin{bmatrix} 5.866025 & 4.048492 \\ 18.330975 & 11.464102 \end{bmatrix} \tag{94}$$

Putting equations (93) and (94) together ⇒

$$T_{\bar{j}}^{\bar{\imath}\bar{k}} = \left[\begin{bmatrix} 4.000000 & 4.188790 \\ 22.918312 & 16.000000 \end{bmatrix} \quad \begin{bmatrix} 10.471976 & 6.579736 \\ 28.000000 & 16.755161 \end{bmatrix}\right] \tag{95}$$

Using equation (95) in the outer product, gives the same results as equation (92).[3]

$$T_{\bar{j}}^{\bar{\imath}\bar{k}} \boldsymbol{e}_{\bar{\imath}} \otimes \boldsymbol{\omega}^{\bar{j}} \otimes \boldsymbol{e}_{\bar{k}} = \left[\begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} \quad \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix}\right] \tag{96}$$

Index Raising and Lowering

$$\boldsymbol{T} = T(i,j,k)\boldsymbol{l}(i) \otimes \boldsymbol{f}(j) \otimes \boldsymbol{h}(k) = [F^T] \otimes [T(i,j,k) \cdot L] \otimes [H]$$

Start with tensor components T_j^{ik} ⇒

$$\boldsymbol{T} = T_j^{ik} \boldsymbol{e_i} \otimes \boldsymbol{\omega^j} \otimes \boldsymbol{e_k} = [W^T] \otimes \left[T_j^{ik} \cdot E\right] \otimes [E] \tag{97}$$

Now raise the 2nd index ⇒

$$\boldsymbol{T} = T^{ijk} \boldsymbol{e_i} \otimes \boldsymbol{e_j} \otimes \boldsymbol{e_k} = [E^T] \otimes \left[T^{ijk} \cdot E\right] \otimes [E] \tag{98}$$

Set equation (97) equal to equation (98) so that the two different configurations resolve to the same result – see [4]

$$[W^T] \otimes \left[T_j^{ik} \cdot E\right] \otimes [E] = [E^T] \otimes \left[T^{ijk} \cdot E\right] \otimes [E] \ \Rightarrow \tag{99}$$

$$[I] \otimes \left[T_j^{ik} \cdot E\right] \otimes [E] = \left[T_j^{ik} \cdot E\right] \otimes [E] = [EE^T] \otimes \left[T^{ijk} \cdot E\right] \otimes [E] = [G] \otimes \left[T^{ijk} \cdot E\right] \otimes [E] \ \Rightarrow$$

$$\left[T_j^{ik} \cdot E\right] = [G] \otimes \left[T^{ijk} \cdot E\right] \ \Rightarrow$$

$$T_j^{ik} = [G] \otimes T^{ijk} \tag{100}$$

---

[3] Tensor Transform Code.

[4] Coordinate Summary - p. 5

Inverting equation (100) ⇒

$$T^{ijk} = [G^{-1}] \otimes T_j^{ik} \tag{101}$$

Note that $' \cdot '$ represents a block vector product.

Now, start with T^{ijk} and lower the 1st index

$$[E^T] \otimes \left[T^{ijk} \cdot E\right] \otimes [E] = [E^T] \otimes \left[T_i^{jk} \cdot W\right] \otimes [E] \;\Rightarrow$$

$$\left[T^{ijk} \cdot E\right] = \left[T_i^{jk} \cdot W\right] \;\Rightarrow$$

$$T_i^{jk} = \left[T_i^{jk} \cdot WE^T\right] = \left[T^{ijk} \cdot EE^T\right] = \left[T^{ijk} \cdot G\right] \;\Rightarrow$$

$$T_i^{jk} = T^{ijk} \cdot G \tag{102}$$

Inverting equation (102) ⇒

$$T^{ijk} = T_i^{jk} \cdot G^{-1} \tag{103}$$

Start with T^{ijk} and lower the 3rd index

$$[E^T] \otimes \left[T^{ijk} \cdot E\right] \otimes [E] = [E^T] \otimes \left[T_k^{ij} \cdot E\right] \otimes [W] \;\Rightarrow$$

$$\left[T^{ijk} \cdot E\right] \otimes [EE^T] = \left[T^{ijk} \cdot E\right] \otimes [G] = \left[T_k^{ij} \cdot E\right] \otimes [WE^T] = \left[T_k^{ij} \cdot E\right] \Rightarrow$$

$$\left[T_k^{ij} \cdot E\right] = \left[T^{ijk} \cdot E\right] \otimes [G] \tag{104}$$

Inverting equation (104) ⇒

$$\left[T^{ijk} \cdot E\right] = \left[T_k^{ij} \cdot E\right] \otimes [G^{-1}] \tag{105}$$

Now for equations (104) and (105), we will show that the metric or inverse metric can operate directly on the tensor coordinates. Consider 2×2 basis matrices ⇒

$$\left[T_k^{ij} \cdot E\right] = \left[T_k^{1j} \quad T_k^{2j}\right] \cdot E = \left[T_k^{1j}E_{11} + T_k^{2j}E_{21} \quad T_k^{1j}E_{12} + T_k^{2j}E_{22}\right] \tag{106}$$

$$\left[T^{ijk} \cdot E\right] = [T^{1jk} \quad T^{2jk}] \cdot E = [T^{1jk}E_{11} + T^{2jk}E_{21} \quad T^{1jk}E_{12} + T^{2jk}E_{22}]$$

$$\left[T^{ijk} \cdot E\right] \otimes [G] = \left[\left[T^{1jk}E_{11} + T^{2jk}E_{21}\right]G \quad \left[T^{1jk}E_{12} + T^{2jk}E_{22}\right]G\right]$$

$= [T^{1jk}E_{11}G + T^{2jk}E_{21}G \quad T^{1jk}E_{12}G + T^{2jk}E_{22}G] = [T^{1jk}GE_{11} + T^{2jk}GE_{21} \quad T^{1jk}GE_{12} + T^{2jk}GE_{22}]$

$= [T^{ijk} \otimes [G] \cdot E] \Rightarrow$

$[T_k^{ij} \cdot E] = [T^{ijk} \cdot E] \otimes [G] = [T^{ijk} \otimes [G] \cdot E] \Rightarrow$

$$T_k^{ij} = T^{ijk} \otimes [G] \tag{107}$$

Inverting equation (107) ⇒

$$T^{ijk} = T_k^{ij} \otimes [G^{-1}] \tag{108}$$

The operations can be combined to lower 2 or 3 indices – see Table 5.

Table 5 shows a summary of the raising and lowering operations for a third order tensor.

Index Lowering	Indices Changed	Index Raising
$T_i^{jk} = T^{ijk} \cdot G$	1st index	$T_{jk}^{i} = T_{ijk} \cdot G^{-1}$
$T_j^{ik} = [G] \otimes T^{ijk}$	2nd index	$T_{ik}^{j} = [G^{-1}] \otimes T_{ijk}$
$T_k^{ij} = T^{ijk} \otimes [G]$	3rd index	$T_{ij}^{k} = T_{ijk} \otimes [G^{-1}]$
$T_{ij}^{k} = [G] \otimes [T^{ijk} \cdot G]$	1st and 2nd index	$T_k^{ij} = [G^{-1}] \otimes [T_{ijk} \cdot G^{-1}]$
$T_{ik}^{j} = [T^{ijk} \cdot G] \otimes [G]$	1st and 3rd index	$T_j^{ik} = [T_{ijk} \cdot G^{-1}] \otimes [G^{-1}]$
$T_{ijk} = [G] \otimes [T^{ijk} \cdot G] \otimes [G]$	1st, 2nd, and 3rd index	$T^{ijk} = [G^{-1}] \otimes [T_{ijk} \cdot G^{-1}] \otimes [G^{-1}]$

Table 5

Note: all these configurations are tested in the software.[5]

4th Order Tensors

Equation (109) is the general equation for a fourth order tensor

$$\boldsymbol{T} = T(i,j,k,l)\boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes \boldsymbol{f}(k) \otimes \boldsymbol{h}(l) \tag{109}$$

where

$\boldsymbol{c}(i),\ \boldsymbol{d}(j),\ \boldsymbol{f}(k),\ \boldsymbol{h}(l) \equiv \begin{Bmatrix} \boldsymbol{e} \\ \boldsymbol{\omega} \end{Bmatrix}$ are bases

Now, create a weighting block matrix - $T_{block}(i,j)$ – as shown in equation (110).

[5] Tensor Transform Code

$$T_{block}(i,j) = T(i,j,k,l)\boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = F^T T(i,j,k,l)H \tag{110}$$

Equation (110) is the same pattern as the second order tensor from equations (15) and (16) where F and H are matrices with row vectors of $\boldsymbol{f}(k)$ and $\boldsymbol{h}(l)$ respectively.

Using equation (110) in equation (109) ⇒

$$\boldsymbol{T} = T(i,j,k,l)\boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes \boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = \boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes T_{block}(i,j) \tag{111}$$

where

> there is a summation over i and j
> $i,j,k,l = 1,n$

To evaluate equation (110), put $\boldsymbol{c}(i) \otimes \boldsymbol{d}(j)$ in index notation ⇒

$$\boldsymbol{c}(i) = [C_{i1} \quad \cdots \quad C_{in}]$$

$$\boldsymbol{d}(j) = [D_{j1} \quad \cdots \quad D_{jn}] \;\Rightarrow$$

$$[\boldsymbol{c}(i) \otimes \boldsymbol{d}(j)]_{\mu\nu} = C_{i\mu}D_{j\nu} \tag{112}$$

where

> C and D are matrices where each row is a basis element.

Now evaluate $\boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes T_{block}(i,j)$ where the summation is over i and j.

For a fixed but arbitrary μ and ν ⇒

$$[\boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes T_{block}(i,j)]_{\mu\nu} = C_{i\mu}D_{j\nu}T_{block}(i,j) = col(C,\mu)T_{block}(i,j)col(D,\nu)$$

$$= row(C^T,\mu)T_{block}(i,j)col(D,\nu) \tag{113}$$

Equation (113) uses regular matrix multiplications except that each element of the block matrix $T_{block}(i,j)$ is a matrix.

Allowing μ, and ν to be arbitrary and not fixed ⇒

$$\boldsymbol{T} = T(i,j,k,l)\boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes \boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = C^T \cdot\; T_{Block}(i,j) \cdot D \tag{114}$$

where

> $'\cdot'$ is a regular matrix multiplication but each submatrix of the block matrix is treated as an element - the same as used for the third order tensor.
>
> $T_{block}(i,j) = T(i,j,k,l)\boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = F^T T(i,j,k,l)H$

Equation (114) is a general equation to evaluate a fourth order tensor.

To show an example of how the $' \cdot '$ works, consider a 2×2 case with $\mu = 1$ and $\nu = 1$

$$[\boldsymbol{T}]_{11} = [[C^T]_{11} \quad [C^T]_{12}] \cdot \begin{bmatrix} T_{Block}(1,1) & T_{Block}(1,2) \\ T_{Block}(2,1) & T_{Block}(2,2) \end{bmatrix} \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

$$= [[C^T]_{11} \quad [C^T]_{12}] \cdot \begin{bmatrix} T_{Block}(1,1)D_{11} + T_{Block}(1,2)D_{21} \\ T_{Block}(2,1)D_{11} + T_{Block}(2,2)D_{21} \end{bmatrix} =$$

$$= [C^T]_{11}[T_{Block}(1,1)D_{11} + T_{Block}(1,2)D_{21}] + [C^T]_{12}[T_{Block}(2,1)D_{11} + T_{Block}(2,2)D_{21}]$$

$$= [C^T]_{11}T_{Block}(1,1)D_{11} + [C^T]_{11}T_{Block}(1,2)D_{21} + [C^T]_{12}T_{Block}(2,1)D_{11} + [C^T]_{12}T_{Block}(2,2)D_{21} \tag{115}$$

Generalizing equation (115) to arbitrary μ and ν $\Rightarrow$

$$[\boldsymbol{T}]_{\mu\nu} = [C^T]_{\mu 1}T_{Block}(1,1)D_{1\nu} + [C^T]_{\mu 1}T_{Block}(1,2)D_{2\nu} + [C^T]_{\mu 2}T_{Block}(2,1)D_{1\nu}$$

$$+[C^T]_{\mu 2}T_{Block}(2,2)D_{2\nu} \tag{116}$$

Making equation (116) completely general $\Rightarrow$

$$[\boldsymbol{T}]_{\mu\nu} = [C^T]_{\mu 1}T_{Block}(i,j)D_{j\nu} \tag{117}$$

Example Fourth Order Tensor Computation

Let

$$T_{jk}^{il} = \begin{bmatrix} \begin{bmatrix} T_{11}^{11} & T_{12}^{11} \\ T_{11}^{12} & T_{12}^{12} \end{bmatrix} & \begin{bmatrix} T_{21}^{11} & T_{22}^{11} \\ T_{21}^{12} & T_{22}^{12} \end{bmatrix} \\ \begin{bmatrix} T_{11}^{21} & T_{12}^{21} \\ T_{11}^{22} & T_{12}^{22} \end{bmatrix} & \begin{bmatrix} T_{21}^{21} & T_{22}^{21} \\ T_{21}^{22} & T_{22}^{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} & \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \end{bmatrix} \tag{118}$$

$$E(r,\theta) = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -r\ sin(\theta) & r\ cos(\theta) \end{bmatrix} \tag{119}$$

$$r = 2$$

$$\theta = \frac{\pi}{3}$$

$$E\left(2, \frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$\boldsymbol{e_1} = [0.5 \quad 0.8660254]$$
$$\boldsymbol{e_2} = [-1.73205081 \quad 1]$$

(120)

(121)

$$W(r,\theta) = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -\dfrac{sin(\theta)}{r} & \dfrac{cos(\theta)}{r} \end{bmatrix}$$

(122)

$$W\left(2,\frac{\pi}{3}\right) = \begin{bmatrix} \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ -\dfrac{\sqrt{3}}{4} & \dfrac{1}{4} \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix}$$

(123)

$$\boldsymbol{\omega^1} = [0.500000 \quad 0.866025]$$
$$\boldsymbol{\omega^2} = [-0.433013 \quad 0.250000]$$

(124)

Equation (125) shows the result of the calculation. For a detailed look at the calculation – see Appendix B

$$T^{il}_{jk}\boldsymbol{e_i} \otimes \boldsymbol{\omega^j} \otimes \boldsymbol{\omega^k} \otimes \boldsymbol{e_l} = \begin{bmatrix} \begin{bmatrix} -0.171187 & 0.268250 \\ -1.334078 & 2.170071 \end{bmatrix} & \begin{bmatrix} 0.183871 & -0.128967 \\ 28.352762 & -40.451176 \end{bmatrix} \\ \begin{bmatrix} 0.331072 & -0.515024 \\ 3.216705 & -5.098377 \end{bmatrix} & \begin{bmatrix} 0.917667 & -1.616326 \\ -23.013998 & 31.083449 \end{bmatrix} \end{bmatrix}$$

(125)

Example Tensor Computed Using Matrix Operations

$$T_{block}(i,j) = T^i_j = T^{il}_{jk}\boldsymbol{\omega^k} \otimes \boldsymbol{e_l}$$

(121)

$$T_{block}{}^1_1 = T^{11}_{11}\boldsymbol{\omega^1} \otimes \boldsymbol{e_1} + T^{12}_{11}\boldsymbol{\omega^1} \otimes \boldsymbol{e_2} + T^{11}_{12}\boldsymbol{\omega^2} \otimes \boldsymbol{e_1} + T^{12}_{12}\boldsymbol{\omega^2} \otimes \boldsymbol{e_2}$$

$$= (1)\boldsymbol{\omega^1} \otimes \boldsymbol{e_1} + (2)\boldsymbol{\omega^1} \otimes \boldsymbol{e_2} + (3)\boldsymbol{\omega^2} \otimes \boldsymbol{e_1} + (\boldsymbol{4})\boldsymbol{\omega^2} \otimes \boldsymbol{e_2}$$

(122)

Table 6 shows the tensor product of the various combinations of components $\boldsymbol{\omega^k} \otimes \boldsymbol{e_l}$

$\otimes$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$
$\boldsymbol{\omega^1}$	$\begin{bmatrix} 0.250000 & 0.433013 \\ 0.433013 & 0.750000 \end{bmatrix}$	$\begin{bmatrix} -0.866025 & 0.500000 \\ -1.500000 & 0.866025 \end{bmatrix}$
$\boldsymbol{\omega^2}$	$\begin{bmatrix} -0.216506 & -0.375000 \\ 0.125000 & 0.216506 \end{bmatrix}$	$\begin{bmatrix} 0.750000 & -0.433013 \\ -0.433013 & 0.250000 \end{bmatrix}$

Table 6

Using Table 6, and equation (122) ⇒

$$T_{block}{}_1^1 = (1)\begin{bmatrix} 0.250000 & 0.433013 \\ 0.433013 & 0.750000 \end{bmatrix} + (2)\begin{bmatrix} -0.866025 & 0.500000 \\ -1.500000 & 0.866025 \end{bmatrix}$$

$$+(3)\begin{bmatrix} -0.216506 & -0.375000 \\ 0.125000 & 0.216506 \end{bmatrix} + (4)\begin{bmatrix} 0.750000 & -0.433013 \\ -0.433013 & 0.250000 \end{bmatrix}$$

$$= \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} \tag{123}$$

The other terms will be computed in the next section using a more streamlined approach.

Streamlined Matrix Approach for Computations

Equation (16) is listed for reference.

$$\boldsymbol{T} = T^{kl}\boldsymbol{e_k^T}\boldsymbol{e_l} = E_{ik}^T T^{kl} E_{lj} = E^T T E \tag{16}$$

Generalizing equation (16) ⇒

$$\boldsymbol{T} = T(k,l)\boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = T(k,l)\boldsymbol{f^T}(k)\boldsymbol{h}(l) = F^T T H \tag{124}$$

Using equation (124) in the T_{block} calculation ⇒

$$T_{block}(i,j) = T(i,j,k,l)\boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = F^T T(i,j,k,l) H \tag{125}$$

Example Streamlined Computations

Using equation (125) for this example ⇒

$$T_{block}{}_1^1 = T_{1k}^{1l}\boldsymbol{\omega^k} \otimes \boldsymbol{e_l} = W^T T_{j1}^{i1} E$$

$$= \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$= \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} \tag{126}$$

Equation (126) is the same as equation (123) as expected.

Calculation of the other $T_{block}{}_j^i$ terms ⇒

$$T_{block}{}_2^1 = W^T T_{2k}^{1l} E = \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$= \begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix}$$

(127)

$$T_{block}{}^{2}_{1} = W^T T^{2l}_{1k} E = \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix} \begin{bmatrix} 9.000000 & 10.000000 \\ 11.000000 & 12.000000 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$= \begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix}$$

(128)

$$T_{block}{}^{2}_{2} = W^T T^{2l}_{2k} E = \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix} \begin{bmatrix} 13.000000 & 14.000000 \\ 15.000000 & 16.000000 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$= \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}$$

(129)

Equations (126) – (129) ⇒

$$T_{block}{}^{i}_{j} = \begin{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} & \begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix} \\ \begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix} & \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \end{bmatrix}$$

(130)

Now use equation (114) to compute the tensor

$C = \boldsymbol{e_i} \Rightarrow E$ and $D = \boldsymbol{\omega^j} \Rightarrow W \;\Rightarrow$

$\boldsymbol{T} = T(i,j,k,l)\boldsymbol{e_i} \otimes \boldsymbol{\omega^j} \otimes \boldsymbol{\omega^k} \otimes \boldsymbol{e_l} = E^T \cdot\, T_{Block}(i,j) \cdot W =$

$$\begin{bmatrix} 0.500000 & -1.732051 \\ 0.866025 & 1.000000 \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} & \begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix} \\ \begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix} & \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix}$$

(131)

Computing each term separately using equation (131) ⇒

$$[\boldsymbol{T}]_{11} = (0.500000) \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} (0.500000)$$

$$+ (0.500000) \begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix} (-0.433013)$$

$$+ (-1.732051) \begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix} (0.500000)$$

$$+(-1.732051)\begin{bmatrix}-0.121951 & 0.075962\\ -20.424038 & 29.121951\end{bmatrix}(-0.433013)=\begin{bmatrix}-0.171187 & 0.268250\\ -1.334078 & 2.170071\end{bmatrix} \tag{132}$$

$$[\boldsymbol{T}]_{12}=(0.500000)\begin{bmatrix}0.868430 & -1.424038\\ -3.924038 & 4.131570\end{bmatrix}(0.866025)$$

$$+(0.500000)\begin{bmatrix}0.538303 & -0.924038\\ -9.424038 & 12.461697\end{bmatrix}(0.250000)$$

$$+(-1.732051)\begin{bmatrix}0.208176 & -0.424038\\ -14.924038 & 20.791824\end{bmatrix}(0.866025)$$

$$+(-1.732051)\begin{bmatrix}-0.121951 & 0.075962\\ -20.424038 & 29.121951\end{bmatrix}(0.250000)=\begin{bmatrix}0.183871 & -0.128967\\ 28.352762 & -40.451176\end{bmatrix} \tag{133}$$

$$[\boldsymbol{T}]_{21}=(0.866025)\begin{bmatrix}0.868430 & -1.424038\\ -3.924038 & 4.131570\end{bmatrix}(0.500000)$$

$$+(0.866025)\begin{bmatrix}0.538303 & -0.924038\\ -9.424038 & 12.461697\end{bmatrix}(-0.433013)$$

$$+(1.000000)\begin{bmatrix}0.208176 & -0.424038\\ -14.924038 & 20.791824\end{bmatrix}(0.500000)$$

$$+(1.000000)\begin{bmatrix}-0.121951 & 0.075962\\ -20.424038 & 29.121951\end{bmatrix}(-0.433013)=\begin{bmatrix}0.331072 & -0.515024\\ 3.216705 & -5.098377\end{bmatrix} \tag{134}$$

$$[\boldsymbol{T}]_{22}=(0.866025)\begin{bmatrix}0.868430 & -1.424038\\ -3.924038 & 4.131570\end{bmatrix}(0.866025)$$

$$+(0.866025)\begin{bmatrix}0.538303 & -0.924038\\ -9.424038 & 12.461697\end{bmatrix}(0.250000)$$

$$+(1.000000)\begin{bmatrix}0.208176 & -0.424038\\ -14.924038 & 20.791824\end{bmatrix}(0.866025)$$

$$+(1.000000)\begin{bmatrix}-0.121951 & 0.075962\\ -20.424038 & 29.121951\end{bmatrix}(0.250000)=\begin{bmatrix}0.917667 & -1.616326\\ -23.013998 & 31.083449\end{bmatrix} \tag{135}$$

Putting equations (132) to (135) into a block matrix ⇒

$$\boldsymbol{T}=T^{il}_{jk}\boldsymbol{e_i}\otimes\boldsymbol{\omega^j}\otimes\boldsymbol{\omega^k}\otimes\boldsymbol{e_l}=\begin{bmatrix}\begin{bmatrix}-0.171187 & 0.268250\\ -1.334078 & 2.170071\end{bmatrix} & \begin{bmatrix}0.183871 & -0.128967\\ 28.352762 & -40.451176\end{bmatrix}\\ \begin{bmatrix}0.331072 & -0.515024\\ 3.216705 & -5.098377\end{bmatrix} & \begin{bmatrix}0.917667 & -1.616326\\ -23.013998 & 31.083449\end{bmatrix}\end{bmatrix} \tag{136}$$

Equation (136) is the same as equation (125) as expected.

Configurations of Fourth Order Tensors

Table 7 show the various configurations for fourth order tensors.

T	H	E	W
F			
E		$T = [T(i,j)]^{kl}$	$T = [T(i,j)]_l^k$
W		$T = [T(i,j)]_k^l$	$T = [T(i,j)]_{kl}$

Table 7

Note:

If $T = T(i,j,k,l)$ $\Rightarrow$ Table 7 generates T_{block} using equation (110).

If $T = T_{block}(i,j)$ $\Rightarrow$ Table 7 generates $\boldsymbol{T}$ from equation (114).

Equation (110) can be inverted and solved for $T(i,j,k,l)$ as shown in equation (137).

$$T_{block}(i,j) = T(i,j,k,l)\boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = F^T T(i,j,k,l)H \;\Rightarrow \tag{110}$$

$$T(i,j,k,l) = [F^T]^{-1} T_{block}(i,j) H^{-1} \tag{137}$$

Table 8 shows the configurations for equation (137).

$T_{block}(i,j)$	H^{-1}	W^T	E^T
$[F^T]^{-1}$			
W		$[T_{block}(i,j)]^{kl}$	$[T_{block}(i,j)]_l^k$
E		$[T_{block}(i,j)]_k^l$	$[T_{block}(i,j)]_{kl}$

Table 8

where

$$WE^T = I = W^T E$$

Transformations of Fourth Order Tensors

A tensor is invariant under a change in coordinates. Equation (114) gives the general equation for a fourth order tensor.

$$\boldsymbol{T} = T(i,j,k,l)\boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes \boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = C^T \cdot\, T_{Block}(i,j) \cdot D \tag{114}$$

The tensor is invariant with respect to changes in coordinates $\Rightarrow$

$$\boldsymbol{T} = C^T \cdot\, T_{Block}(i,j) \cdot D = \bar{C}^T \cdot\, T_{Block}(\bar{\imath},\bar{\jmath}) \cdot \bar{D} \tag{138}$$

The transformed coordinate relations are as follows:

$\bar{C} = M_1 C$

$\bar{D} = M_2 D$

(139)

where

$$M_1, M_2 = \begin{cases} A \; for \; basis \; E \\ B^T \; for \; reciprocal \; basis \; W \end{cases}$$

Using equation (139) in equation (138) ⇒

$\boldsymbol{T} = C^T \cdot T_{Block}(i,j) \cdot D = \bar{C}^T \cdot T_{Block}(\bar{\imath},\bar{\jmath}) \cdot \bar{D} = (M_1 C)^T \, T_{Block}(\bar{\imath},\bar{\jmath}) M_2 D = C^T M_1^T T_{Block}(\bar{\imath},\bar{\jmath}) M_2 D \;\Rightarrow$

$T_{Block}(i,j) = M_1^T T_{Block}(\bar{\imath},\bar{\jmath}) M_2$

(140)

Solving equation (140) for $T_{Block}(\bar{\imath},\bar{\jmath}) \;\Rightarrow$

$T_{Block}(\bar{\imath},\bar{\jmath}) = [M_1^T]^{-1} T(i,j) [M_2]^{-1} = ([M_1]^{-1})^T T(i,j) [M_2]^{-1}$

(141)

Table 9 shows the combinations for equation (141).

$T(i,j)$	$[M_2]^{-1}$	$A^{-1} = B$	$[B^T]^{-1} = A^T$
$[M_1]^{-1}$			
$[A]^{-1} = B$		$T(u,u) = T^{ij}$	$T(u,d) = T_j^i$
$B^{-1} = A$		$T(d,u) = T_i^j$	$T(d,d) = T_{ij}$

Table 9

Fourth Order Tensor Transform Example

The coordinate conversion equations for the polar based transform are given by equation (142).

$r = \sqrt{\bar{r}}$
$\theta = \sqrt{\bar{\theta}}$

$\bar{r} = r^2$
$\bar{\theta} = \theta^2$

(142)

The transformation matrices are given by equations (143) and (144) respectively.

$$A = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{143}$$

$$B = \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix} \tag{144}$$

The transformation equations are given below

$\bar{E} = AE$

$\bar{W} = B^T W$

For $r = 2$ and $\theta = \frac{\pi}{3}$ $\Rightarrow$

$\bar{r} = 4$

$\bar{\theta} = \frac{\pi^2}{9}$ $\Rightarrow$

From earlier example ⇒

$$A = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix}$$

$$B = \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix}$$

$$E = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$W = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix}$$

$$\bar{E} = AE = \begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} = \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix}$$

$$\bar{W} = B^T W \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix} = \begin{bmatrix} 2.000000 & 3.464102 \\ -0.906900 & 0.523599 \end{bmatrix}$$

${T_{block}}_j^i$ from the previous example is shown - equation (145).

$$T_{block}{}^{i}_{j} = \begin{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} & \begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix} \\ \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix} & \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \end{bmatrix} \tag{145}$$

From Table 9, equation (141) ⇒

$$T_{block}{}^{\bar{\imath}}_{\bar{\jmath}} = ([M_1]^{-1})^T T(i,j)[M_2]^{-1} = B^T T_{block}{}^{i}_{j} A^T =$$

$$\begin{bmatrix} 0.250000 & 0.000000 \\ 0.000000 & 0.477465 \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} & \begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix} \\ \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix} & \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} 4.000000 & 0.000000 \\ 0.000000 & 2.094395 \end{bmatrix} \tag{146}$$

Computing equation (146) term by term ⇒

$$\begin{aligned} T_{block}{}^{\bar{1}}_{\bar{1}} &= (4.000000)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(0.250000) \\ &+(4.000000)\begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix}(0.000000) \\ &+(0.000000)\begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix}(0.250000) \\ &+(0.000000)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(0.000000) \\ &= (4.000000)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(0.250000) = \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} \end{aligned} \tag{147}$$

$$\begin{aligned} T_{block}{}^{\bar{1}}_{\bar{2}} &= (4.000000)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(0.000000) \\ &+(4.000000)\begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix}(0.477465) \\ &+(0.000000)\begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix}(0.000000) \\ &+(0.000000)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(0.477465) \\ &= (4.000000)\begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix}(0.477465) = \begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix} \end{aligned} \tag{148}$$

$$T_{block}{}^{\bar{2}}_{\bar{1}} = (0.000000)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(0.250000)$$

$$+(0.000000)\begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix}(0.000000)$$

$$+(2.094395)\begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix}(0.250000)$$

$$+(2.094395)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(0.000000)$$

$$= (2.094395)\begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix}(0.250000) = \begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix} \tag{149}$$

$$T_{block}{}^{\bar{2}}_{\bar{2}} = (0.000000)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(0.000000)$$

$$+(0.000000)\begin{bmatrix} 0.538303 & -0.924038 \\ -9.424038 & 12.461697 \end{bmatrix}(0.477465)$$

$$+(2.094395)\begin{bmatrix} 0.208176 & -0.424038 \\ -14.924038 & 20.791824 \end{bmatrix}(0.000000)$$

$$+(2.094395)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(0.477465)$$

$$= (2.094395)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(0.477465) = \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \tag{150}$$

Putting equations (147) – (150) together ⇒

$$T_{block}{}^{\bar{\iota}}_{\bar{J}} = \begin{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} & \begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix} \\ \begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix} & \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \end{bmatrix} \tag{151}$$

Compute tensor using the inner product – equation (138) ⇒

$$\boldsymbol{T} = T^{\bar{\iota}\bar{l}}_{\bar{J}\bar{k}} \boldsymbol{e}_{\bar{\iota}} \otimes \boldsymbol{\omega}^{\bar{J}} \otimes \boldsymbol{\omega}^{\bar{k}} \otimes \boldsymbol{e}_{\bar{l}} = \bar{E} \cdot T_{block}{}^{\bar{\iota}}_{\bar{J}} \cdot \bar{W}$$

$$= \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} & \begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix} \\ \begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix} & \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \end{bmatrix}$$

$$\cdot \begin{bmatrix} 2.000000 & 3.464102 \\ -0.906900 & 0.523599 \end{bmatrix} \tag{152}$$

Now evaluate equation (152) term by term ⇒

$$[\boldsymbol{T}]_{\bar{1}\,\bar{1}} = (0.125000)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(2.000000)$$

$$+(0.125000)\begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix}(-0.906900)$$

$$+(-0.826993)\begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix}(2.000000)$$

$$+(-0.826993)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(-0.906900)$$

$$= \begin{bmatrix} -0.171187 & 0.268250 \\ -1.334078 & 2.170071 \end{bmatrix} \tag{153}$$

$$[\boldsymbol{T}]_{\bar{1}\,\bar{2}} = (0.125000)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(3.464102)$$

$$+(0.125000)\begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix}(0.523599)$$

$$+(-0.826993)\begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix}(3.464102)$$

$$+(-0.826993)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(0.523599)$$

$$= \begin{bmatrix} 0.183871 & -0.128967 \\ 28.352762 & -40.451176 \end{bmatrix} \tag{154}$$

$$[\boldsymbol{T}]_{\bar{2}\,\bar{1}} = (0.216506)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(2.000000)$$

$$+(0.216506)\begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix}(-0.906900)$$

$$+(0.477465)\begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix}(2.000000)$$

$$+(0.477465)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(-0.906900)$$

$$= \begin{bmatrix} 0.331072 & -0.515024 \\ 3.216705 & -5.098377 \end{bmatrix} \tag{155}$$

$$[\boldsymbol{T}]_{\bar{2}\,\bar{2}} = (0.216506)\begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix}(3.464102)$$

$$+(0.216506)\begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix}(0.523599)$$

$$+(0.477465)\begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix}(3.464102)$$

$$+(0.477465)\begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix}(0.523599)$$

$$= \begin{bmatrix} 0.917667 & -1.616326 \\ -23.013998 & 31.083449 \end{bmatrix}$$

(156)

Putting equations (153) – (156) together ⇒

$$\boldsymbol{T} = T_{\bar{j}\,\bar{k}}^{\bar{\imath}\,\bar{l}} \boldsymbol{e}_{\bar{\imath}} \otimes \boldsymbol{\omega}^{\bar{j}} \otimes \boldsymbol{\omega}^{\bar{k}} \otimes \boldsymbol{e}_{\bar{l}} = \begin{bmatrix} \begin{bmatrix} -0.171187 & 0.268250 \\ -1.334078 & 2.170071 \end{bmatrix} & \begin{bmatrix} 0.183871 & -0.128967 \\ 28.352762 & -40.451176 \end{bmatrix} \\ \begin{bmatrix} 0.331072 & -0.515024 \\ 3.216705 & -5.098377 \end{bmatrix} & \begin{bmatrix} 0.917667 & -1.616326 \\ -23.013998 & 31.083449 \end{bmatrix} \end{bmatrix}$$

(157)

Equation (157) is the same as equation (136) as expected.

Transform $T_{\bar{j}}^{\bar{\imath}} \rightarrow T_{\bar{j}\,\bar{k}}^{\bar{\imath}\,\bar{l}}$

First show the transformation of one submatrix in the unprimed system.

Equation (142) was derived in a previous section and shows the transform of $T_{block}(i,j)$ to $T(i,j,k,l)$.

$$T(i,j,k,l) = [F^T]^{-1} T_{block}(i,j) H^{-1} = [F^{-1}]^T T_{block}(i,j) H^{-1}$$

(142)

From equation (142) and Table 9 ⇒

T_{jk}^{il} is a case in Table 9 as $[T_{block}(i,j)]_k^l \;\Rightarrow\; [F^T]^{-1} = E,\;\; H^{-1} = W^T$

Equation (142) ⇒

$$T_{jk}^{il} = E T_j^i W^T$$

(158)

$$W^T = \begin{bmatrix} 0.500000 & -0.433013 \\ 0.866025 & 0.250000 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$W = [E^{-1}]^T = \begin{bmatrix} 0.50000000 & 0.86602540 \\ -0.43301270 & 0.25000000 \end{bmatrix}$$

(159)

$$T^{ij11} = W T^{11} W^T$$

$$= \begin{bmatrix} 0.50000000 & 0.86602540 \\ -0.43301270 & 0.25000000 \end{bmatrix} \begin{bmatrix} 7.91987298 & -9.99519053 \\ -7.99519053 & 9.08012702 \end{bmatrix} \begin{bmatrix} 0.50000000 & -0.43301270 \\ 0.86602540 & 0.25000000 \end{bmatrix}$$

$$= \begin{bmatrix} 1.00000000 & 2.00000000 \\ 3.00000000 & 4.00000000 \end{bmatrix} \tag{160}$$

Equation (160) is the same as T^{ij11} of the previous example – equation (113).

Now, use equation (142) and Table 9 in the primed coordinate system ⇒

$$T(\bar{\imath},\bar{\jmath},\bar{k},\bar{l}) = [\bar{F}^{-1}]^T T_{block}(\bar{\imath},\bar{\jmath})\bar{H}^{-1} \tag{161}$$

For this example, ⇒

$$T_{block}{}^{\bar{\imath}}_{\bar{\jmath}} = \begin{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} & \begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix} \\ \begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix} & \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \end{bmatrix} \tag{165}$$

Now $T(\bar{\imath},\bar{\jmath},\bar{k},\bar{l}) = T^{\bar{\imath}\,\bar{l}}_{\bar{\jmath}\,\bar{k}}$

This is a case in Table 9 as $[T_{block}(\bar{\imath},\bar{\jmath})]^{\bar{l}}_{\bar{k}} \;\Rightarrow\; [\bar{F}^T]^{-1} = \bar{E},\; H^{-1} = W^T \Rightarrow$

$$T^{\bar{\imath}\,\bar{l}}_{\bar{\jmath}\,\bar{k}} = \bar{E} T_{block}{}^{\bar{\imath}}_{\bar{\jmath}} \bar{W}^T \tag{166}$$

$$\bar{E} = AE = \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix}$$

$$\bar{W} = B^T W = \begin{bmatrix} 2.000000 & 3.464102 \\ -0.906900 & 0.523599 \end{bmatrix} \tag{167}$$

$$T^{\bar{1}\,\bar{l}}_{\bar{1}\,\bar{k}} = \bar{E} T_{block}{}^{\bar{1}}_{\bar{1}} \bar{W}^T = \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix} \begin{bmatrix} 0.868430 & -1.424038 \\ -3.924038 & 4.131570 \end{bmatrix} \begin{bmatrix} 2.000000 & -0.906900 \\ 3.464102 & 0.523599 \end{bmatrix}$$

$$= \begin{bmatrix} 1.000000 & 1.047198 \\ 5.729578 & 4.000000 \end{bmatrix} \tag{168}$$

$$T^{\bar{1}\,\bar{l}}_{\bar{2}\,\bar{k}} = \bar{E} T_{block}{}^{\bar{1}}_{\bar{2}} \bar{W}^T$$

$$= \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix} \begin{bmatrix} 1.028083 & -1.764783 \\ -17.998587 & 23.800088 \end{bmatrix} \begin{bmatrix} 2.000000 & -0.906900 \\ 3.464102 & 0.523599 \end{bmatrix}$$

$$= \begin{bmatrix} 9.549297 & 6.000000 \\ 25.532938 & 15.278875 \end{bmatrix} \tag{169}$$

$$T_{\bar{1}\;\bar{k}}^{\bar{2}\;\bar{l}} = \bar{E}T_{block}{}_{\bar{1}}^{\bar{2}}\bar{W}^T$$

$$= \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix} \begin{bmatrix} 0.109001 & -0.222026 \\ -7.814208 & 10.886574 \end{bmatrix} \begin{bmatrix} 2.000000 & -0.906900 \\ 3.464102 & 0.523599 \end{bmatrix}$$

$$= \begin{bmatrix} 4.712389 & 2.741557 \\ 11.000000 & 6.283185 \end{bmatrix}$$

(170)

$$T_{\bar{2}\;\bar{k}}^{\bar{2}\;\bar{l}} = \bar{E}T_{block}{}_{\bar{2}}^{\bar{2}}\bar{W}^T$$

$$= \begin{bmatrix} 0.125000 & 0.216506 \\ -0.826993 & 0.477465 \end{bmatrix} \begin{bmatrix} -0.121951 & 0.075962 \\ -20.424038 & 29.121951 \end{bmatrix} \begin{bmatrix} 2.000000 & -0.906900 \\ 3.464102 & 0.523599 \end{bmatrix}$$

$$= \begin{bmatrix} 13.000000 & 7.330383 \\ 28.647890 & 16.000000 \end{bmatrix}$$

(171)

Putting equations (168) – (171) together ⇒

$$T_{\bar{j}\;\bar{k}}^{\bar{\imath}\;\bar{l}} = \begin{bmatrix} \begin{bmatrix} 1.000000 & 1.047198 \\ 5.729578 & 4.000000 \end{bmatrix} & \begin{bmatrix} 9.549297 & 6.000000 \\ 25.532938 & 15.278875 \end{bmatrix} \\ \begin{bmatrix} 4.712389 & 2.741557 \\ 11.000000 & 6.283185 \end{bmatrix} & \begin{bmatrix} 13.000000 & 7.330383 \\ 28.647890 & 16.000000 \end{bmatrix} \end{bmatrix}$$

(172)

Now compute the outer product in the primed coordinate system ⇒

$$\boldsymbol{T} = T_{\bar{j}\;\bar{k}}^{\bar{\imath}\;\bar{l}}\boldsymbol{e}_{\bar{\imath}} \otimes \boldsymbol{\omega}^{\bar{j}} \otimes \boldsymbol{\omega}^{\bar{k}} \otimes \boldsymbol{e}_{\bar{l}} = \begin{bmatrix} \begin{bmatrix} -0.171187 & 0.268250 \\ -1.334078 & 2.170071 \end{bmatrix} & \begin{bmatrix} 0.183871 & -0.128967 \\ 28.352762 & -40.451176 \end{bmatrix} \\ \begin{bmatrix} 0.331072 & -0.515024 \\ 3.216705 & -5.098377 \end{bmatrix} & \begin{bmatrix} 0.917667 & -1.616326 \\ -23.013998 & 31.083449 \end{bmatrix} \end{bmatrix}$$

(173)

The calculations have not been shown for equation (172) but the results can be obtained by running the software.[6] Equation (173) is the same as equation (157) as expected.

Index Raising and Lowering

Equation (125) gives the equation for $T_{block}(i,j)$.

$$T_{block}(i,j) = T(i,j,k,l)\boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = F^T T(i,j,k,l)H$$

(125)

The formulation should be changed to explicitly put all the parameters in a single equation. The first step is to put equation (125) in a more general form as shown in equation (174).

[6] Tensor Transform Code

$$T_{block}(i,j) = T(i,j,k,l)\boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = \left[F^T\right] \otimes T(i,j,k,l) \otimes [H] \tag{174}$$

This is the same approach used for third order tensors. Now the equation for the full tensor is given in equation (175).

$$\begin{aligned}\boldsymbol{T} &= T(i,j,k,l)\boldsymbol{c}(i) \otimes \boldsymbol{d}(j) \otimes \boldsymbol{f}(k) \otimes \boldsymbol{h}(l) = C^T \cdot\ T_{Block}(i,j) \cdot D \\ &= C^T \cdot \left[[F^T] \otimes T(i,j,k,l) \otimes [H]\right] \cdot D\end{aligned} \tag{175}$$

For the configuration T_{jk}^{il}

$$C = E, D = W, F = W, H = E \Rightarrow$$

$$\boldsymbol{T} = E^T \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W \tag{176}$$

If we want to lower the first index of T_{jk}^{il} , we get a configuration of T_{ijk}^{l}, Table 7 ⇒

$$C = W, D = W, F = W, H = E \Rightarrow$$

$$\boldsymbol{T} = W^T \cdot \left[[W^T] \otimes T_{ijk}^{l} \otimes [E]\right] \cdot W \tag{177}$$

To simplify equation (177), we will show the following:

$$G \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] = [W^T] \otimes \left[G \cdot T_{jk}^{il}\right] \otimes [E]$$

From the earlier definition ⇒

$$T_{jk}^{il} = \begin{bmatrix} \begin{bmatrix} T_{11}^{11} & T_{12}^{11} \\ T_{11}^{12} & T_{12}^{12} \end{bmatrix} & \begin{bmatrix} T_{21}^{11} & T_{22}^{11} \\ T_{21}^{12} & T_{22}^{12} \end{bmatrix} \\ \begin{bmatrix} T_{11}^{21} & T_{12}^{21} \\ T_{11}^{22} & T_{12}^{22} \end{bmatrix} & \begin{bmatrix} T_{21}^{21} & T_{22}^{21} \\ T_{21}^{22} & T_{22}^{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} T_{1k}^{1l} & T_{2k}^{1l} \\ T_{1k}^{2l} & T_{2k}^{2l} \end{bmatrix} \Rightarrow \tag{178}$$

$$G \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] = G \cdot \begin{bmatrix} W^T T_{1k}^{1l} E & \cdots & W^T T_{nk}^{1l} E \\ \vdots & \ddots & \vdots \\ W^T T_{1k}^{nl} E & \cdots & W^T T_{nk}^{nl} E \end{bmatrix} \tag{179}$$

For indices μ and ν, equation (179) ⇒

$$\left[G \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right]\right]_{\mu\nu} = G_{\mu\alpha} W^T T_{\nu k}^{\alpha l} E \tag{180}$$

For a given μ, α, $G_{\mu\alpha}$ is a constant ⇒

$$G_{\mu\alpha} W^T T_{\nu k}^{\alpha l} E = W^T G_{\mu\alpha} T_{\nu k}^{\alpha l} \;\Rightarrow$$

But μ and ν are arbitrary ⇒

$$G \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] = [W^T] \otimes \left[G \cdot T_{jk}^{il}\right] \otimes [E] \tag{181}$$

In general,

$$M_1 \cdot \left[[M_2] \otimes T\right] = [M_2] \otimes [M_1 \cdot T] \tag{182}$$

where

M_1, and M_2, are matrices
T is a block matrix

The tensor $\boldsymbol{T}$ needs to be invariant, so set equation (176) equal to (177) ⇒

$$E^T \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W = W^T \cdot \left[[W^T] \otimes T_{ijk}^{l} \otimes [E]\right] \cdot W \;\Rightarrow$$

$$\left[[W^T] \otimes T_{ijk}^{l} \otimes [E]\right] \cdot W = EE^T \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W = G \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W \;\Rightarrow \tag{183}$$

Using equation (182) on the lefthand G of equation (183) ⇒

$$\left[[W^T] \otimes T_{ijk}^{l} \otimes [E]\right] \cdot W = G \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W = \left[[W^T] \otimes \left[G \cdot T_{jk}^{il}\right] \otimes [E]\right] \cdot W \;\Rightarrow \tag{184}$$

Using equation (182) again on W on the right-hand side of equation (184) ⇒

$$[W^T] \otimes \left[T_{ijk}^{l} \cdot W\right] \otimes [E] = [W^T] \otimes \left[G \cdot T_{jk}^{il} \cdot W\right] \otimes [E] \;\Rightarrow$$

$$T_{ijk}^{l} = G \cdot T_{jk}^{il} \tag{185}$$

Inverting equation (185) to raise the first index ⇒

$$T_{jk}^{il} = G^{-1} \cdot T_{ijk}^{l} \tag{186}$$

To raise the second index in T_{jk}^{il} to get T_{k}^{ijl}

$$\boldsymbol{T} = E^T \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W = E^T \cdot \left[[W^T] \otimes T_{k}^{ijl} \otimes [E]\right] \cdot E \;\Rightarrow$$

$$\boldsymbol{T} = [W^T] \otimes \left[E^T \cdot T_{jk}^{il} \cdot W\right] \otimes [E] = [W^T] \otimes \left[E^T \cdot T_{k}^{ijl} \cdot E\right] \otimes [E] \;\Rightarrow$$

$$T_{jk}^{il} \cdot W = T_{k}^{ijl} \cdot E \;\Rightarrow$$

$$T_{k}^{ijl} = T_{jk}^{il} \cdot WW^T = T_{jk}^{il} \cdot G^{-1} \tag{187}$$

Inverting equation (187) to lower the second index ⇒

$$T_{jk}^{il} = T_{k}^{ijl} \cdot G \tag{188}$$

To raise the third index in T_{jk}^{il} to get T_{j}^{ikl} ⇒

$$\boldsymbol{T} = E^T \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W = E^T \cdot \left[[E^T] \otimes T_{j}^{ikl} \otimes [E]\right] \cdot W \;\Rightarrow$$

$$\left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] = \left[[E^T] \otimes T_{j}^{ikl} \otimes [E]\right] \;\Rightarrow$$

$$\left[[WW^T] \otimes T_{jk}^{il} \otimes [E]\right] = T_{j}^{ikl} \otimes [E] \;\Rightarrow$$

$$\left[[G^{-1}] \otimes T_{jk}^{il} \otimes [E]\right] = T_{j}^{ikl} \otimes [E] \;\Rightarrow$$

$$T_{j}^{ikl} = [G^{-1}] \otimes T_{jk}^{il} \tag{189}$$

Inverting equation (189) to lower the third index ⇒

$$T_{jk}^{il} = [G] \otimes T_{j}^{ikl} \tag{190}$$

To lower the fourth index in T_{jk}^{il} to get T_{jkl}^{i} ⇒

$$\boldsymbol{T} = E^T \cdot \left[[W^T] \otimes T_{jk}^{il} \otimes [E]\right] \cdot W = E^T \cdot \left[[W^T] \otimes T_{jkl}^{i} \otimes [W]\right] \cdot W \;\Rightarrow$$

$$[W^T] \otimes T_{jk}^{il} \otimes [E] = [W^T] \otimes T_{jkl}^{i} \otimes [W] \;\Rightarrow$$

$$T_{jk}^{il} \otimes [E] = T_{jkl}^{i} \otimes [W] \;\Rightarrow$$

$$T_{jkl}^{i} = T_{jk}^{il} \otimes [EE^T] = T_{jk}^{il} \otimes [G] \tag{191}$$

Inverting equation (191) to raise the fourth index ⇒

$$T^{il}_{jk} = T^{i}_{jkl} \otimes [G^{-1}] \tag{192}$$

These raising and lowering operations can be put together to raise or lower multiple indices.

Table 10 shows a summary of the raising and lowering operations for a fourth order tensor.

Index Lowering	Indices Affected	Index Raising
$T_i^{\ jkl} = G \cdot T^{ijkl}$	1st	$T^{il}_{jk} = G^{-1} \cdot T^{l}_{ijk}$
$T_j^{ikl} = T^{ijkl} \cdot G$	2nd	$T^{ijl}_{k} = T^{il}_{jk} \cdot G^{-1}$
$T_k^{ijl} = [G] \otimes T^{ijkl}$	3rd	$T^{ikl}_{j} = [G^{-1}] \otimes T^{il}_{jk}$
$T_l^{ijk} = T^{ijkl} \otimes [G]$	4th	$T^{il}_{jk} = T^{i}_{jkl} \otimes [G^{-1}]$
$T^{kl}_{ij} = G \cdot T^{ijkl} \cdot G$	1st and 2nd	$T^{ijkl} = G^{-1} \cdot T^{kl}_{ij} \cdot G^{-1}$
$T^{jl}_{ik} = [G] \otimes \left[G \cdot T^{ijkl}\right]$	1st and 3rd	$T^{ijkl} = [G^{-1}] \otimes \left[G^{-1} \cdot T^{jl}_{ik}\right]$
$T^{j}_{ikl} = \left[G \cdot T^{ijkl}\right] \otimes [G]$	1st and 4th	$T^{ijkl} = G^{-1} \cdot T^{j}_{ikl} \otimes [G^{-1}]$
$T^{il}_{jk} = [G] \otimes \left[T^{ijkl} \cdot G\right]$	2nd and 3rd	$T^{ijkl} = [G^{-1}] \otimes T^{il}_{jk} \cdot G^{-1}$
$T^{ij}_{kl} = [G] \otimes T^{ijkl} \otimes [G]$	3rd and 4th	$T^{ijkl} = [G^{-1}] \otimes T^{ij}_{kl} \otimes [G^{-1}]$
$T^{l}_{ijk} = [G] \otimes \left[G \cdot T^{ijkl} \cdot G\right]$	1st, 2nd and 3rd	$T^{ijkl} = [G^{-1}] \otimes \left[G^{-1} \cdot T^{l}_{ijk} \cdot G^{-1}\right]$
$T^{i}_{jkl} = [G] \otimes \left[T^{ijkl} \cdot G\right] \otimes [G]$	2nd, 3rd, and 4th	$[G^{-1}] \otimes \left[T^{ijkl} \cdot G^{-1}\right] \otimes [G^{-1}]$
$T^{j}_{ikl} = [G] \otimes \left[G \cdot T^{ijkl}\right] \otimes [G]$	1st,3rd, and 4th	$[G^{-1}] \otimes \left[G^{-1} \cdot T^{j}_{ikl}\right] \otimes [G^{-1}]$
$T_{ijkl} = [G] \otimes \left[G \cdot T^{ijkl} \cdot G\right] \otimes [G]$	1st,2nd,3rd, and 4th	$T^{ijkl} = [G] \otimes \left[G \cdot T_{ijkl} \cdot G\right] \otimes [G]$

Table 10

We will show an example of how the index raising expressions were reached to avoid confusion.

$$T_{ijkl} = [G] \otimes \left[G \cdot T^{ijkl} \cdot G\right] \otimes [G] \ \Rightarrow$$

$$[G^{-1}] \otimes T_{ijkl} = \left[G \cdot T^{ijkl} \cdot G\right] \otimes [G] \ \Rightarrow$$

$$[G^{-1}] \otimes T_{ijkl} \otimes [G^{-1}] = \left[G \cdot T^{ijkl} \cdot G\right] \ \Rightarrow$$

$$G^{-1} \cdot \left[[G^{-1}] \otimes T_{ijkl} \otimes [G^{-1}]\right] \cdot G^{-1} = T^{ijkl} \tag{193}$$

But the $'\otimes'$ and $'\cdot'$ operators can be interchanged ⇒

$$T^{ijkl} = [G^{-1}] \otimes \left[G^{-1} \cdot T_{ijkl} \cdot G^{-1}\right] \otimes [G^{-1}] \tag{194}$$

Equation (194) shows that to invert an expression from the right column of the table, replace G by G^{-1} and interchange the tensor components.

Example of the Operator Interchange of $'\otimes'$ and $'\cdot'$

$$M_1 = W,\, M_2 = E,\, T = T_{jk}^{il} = \begin{bmatrix} \begin{bmatrix} T_{11}^{11} & T_{12}^{11} \\ T_{11}^{12} & T_{12}^{12} \end{bmatrix} & \begin{bmatrix} T_{21}^{11} & T_{22}^{11} \\ T_{21}^{12} & T_{22}^{12} \end{bmatrix} \\ \begin{bmatrix} T_{11}^{21} & T_{12}^{21} \\ T_{11}^{22} & T_{12}^{22} \end{bmatrix} & \begin{bmatrix} T_{21}^{21} & T_{22}^{21} \\ T_{21}^{22} & T_{22}^{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} & \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \end{bmatrix}$$

Evaluate $W \cdot \big[[E] \otimes T\big]$

$$E\left(2, \frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix}$$

$$W\left(2, \frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix}$$

$$\big[[E] \otimes T\big] = \left[\begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} \otimes \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} & \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \end{bmatrix} \right]$$

$$= \begin{bmatrix} \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} & \begin{bmatrix} 0.500000 & 0.866025 \\ -1.732051 & 1.000000 \end{bmatrix} \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 3.098075 & 4.464100 \\ 1.267949 & 0.535898 \end{bmatrix} & \begin{bmatrix} 8.562175 & 9.928200 \\ -1.660255 & -2.392306 \end{bmatrix} \\ \begin{bmatrix} 14.026275 & 15.392300 \\ -4.588459 & -5.320510 \end{bmatrix} & \begin{bmatrix} 19.490375 & 20.856400 \\ -7.516663 & -8.248714 \end{bmatrix} \end{bmatrix} \tag{195}$$

$$W \cdot \big[[E] \otimes T\big] = \begin{bmatrix} 0.500000 & 0.866025 \\ -0.433013 & 0.250000 \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} 3.098075 & 4.464100 \\ 1.267949 & 0.535898 \end{bmatrix} & \begin{bmatrix} 8.562175 & 9.928200 \\ -1.660255 & -2.392306 \end{bmatrix} \\ \begin{bmatrix} 14.026275 & 15.392300 \\ -4.588459 & -5.320510 \end{bmatrix} & \begin{bmatrix} 19.490375 & 20.856400 \\ -7.516663 & -8.248714 \end{bmatrix} \end{bmatrix} \tag{196}$$

Evaluating equation (196) term by term ⇒

$$\Big[W \cdot \big[[E] \otimes T\big]\Big]_{11} = (0.500000) \begin{bmatrix} 3.098075 & 4.464100 \\ 1.267949 & 0.535898 \end{bmatrix} + (0.866025) \begin{bmatrix} 14.026275 & 15.392300 \\ -4.588459 & -5.320510 \end{bmatrix}$$

$$= \begin{bmatrix} 13.696142 & 15.562167 \\ -3.339746 & -4.339746 \end{bmatrix} \tag{197}$$

$$\left[W \cdot \left[[E] \otimes T\right]\right]_{12}$$

$$= (0.500000) \begin{bmatrix} 8.562175 & 9.928200 \\ -1.660255 & -2.392306 \end{bmatrix} + (0.866025) \begin{bmatrix} 19.490375 & 20.856400 \\ -7.516663 & -8.248714 \end{bmatrix}$$

$$= \begin{bmatrix} 21.160240 & 23.026264 \\ -7.339746 & -8.339746 \end{bmatrix} \tag{198}$$

$$\left[W \cdot \left[[E] \otimes T\right]\right]_{21} = (-0.433013) \begin{bmatrix} 3.098075 & 4.464100 \\ 1.267949 & 0.535898 \end{bmatrix} + (0.250000) \begin{bmatrix} 14.026275 & 15.392300 \\ -4.588459 & -5.320510 \end{bmatrix}$$

$$= \begin{bmatrix} 2.165062 & 1.915062 \\ -1.696153 & -1.562178 \end{bmatrix} \tag{199}$$

$$\left[W \cdot \left[[E] \otimes T\right]\right]_{22}$$

$$= (-0.433013) \begin{bmatrix} 8.562175 & 9.928200 \\ -1.660255 & -2.392306 \end{bmatrix} + (0.250000) \begin{bmatrix} 19.490375 & 20.856400 \\ -7.516663 & -8.248714 \end{bmatrix}$$

$$= \begin{bmatrix} 1.165061 & 0.915060 \\ -1.160254 & -1.026279 \end{bmatrix} \tag{200}$$

Putting equations 197 – 200 together ⇒

$$W \cdot \left[[E] \otimes T\right] = \begin{bmatrix} \begin{bmatrix} 13.696142 & 15.562167 \\ -3.339746 & -4.339746 \end{bmatrix} & \begin{bmatrix} 21.160240 & 23.026264 \\ -7.339746 & -8.339746 \end{bmatrix} \\ \begin{bmatrix} 2.165062 & 1.915062 \\ -1.696153 & -1.562178 \end{bmatrix} & \begin{bmatrix} 1.165061 & 0.915060 \\ -1.160254 & -1.026279 \end{bmatrix} \end{bmatrix} \tag{201}$$

Now evaluate $[E] \otimes [W \cdot T]$

$$[W \cdot T] = \begin{bmatrix} \begin{bmatrix} 8.294225 & 9.660250 \\ 11.026275 & 12.392300 \end{bmatrix} & \begin{bmatrix} 13.758325 & 15.124350 \\ 16.490375 & 17.856400 \end{bmatrix} \\ \begin{bmatrix} 1.816987 & 1.633974 \\ 1.450961 & 1.267948 \end{bmatrix} & \begin{bmatrix} 1.084935 & 0.901922 \\ 0.718909 & 0.535896 \end{bmatrix} \end{bmatrix} \tag{202}$$

$$[E] \otimes [W \cdot T] = \begin{bmatrix} \begin{bmatrix} 13.696142 & 15.562167 \\ -3.339746 & -4.339746 \end{bmatrix} & \begin{bmatrix} 21.160240 & 23.026264 \\ -7.339746 & -8.339746 \end{bmatrix} \\ \begin{bmatrix} 2.165062 & 1.915062 \\ -1.696153 & -1.562178 \end{bmatrix} & \begin{bmatrix} 1.165061 & 0.915060 \\ -1.160254 & -1.026279 \end{bmatrix} \end{bmatrix} \tag{203}$$

Equation (203) is the same as equation (201) ⇒

$W \cdot \big[[E] \otimes T\big] = [E] \otimes [W \cdot T]$

Appendix A -Third Order Tensor Computed Using the Outer Product

$$\boldsymbol{T} = T^i_{jk}\boldsymbol{e_i} \otimes \omega^j \otimes \boldsymbol{e_k} = T^1_{11}\boldsymbol{e_1} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_1} + T^1_{12}\boldsymbol{e_1} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_2} + T^1_{21}\boldsymbol{e_1} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_1}$$

$$+T^1_{22}\boldsymbol{e_1} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_2} + T^2_{11}\boldsymbol{e_2} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_1} + T^2_{12}\boldsymbol{e_2} \otimes \boldsymbol{\omega^1} \otimes \boldsymbol{e_2} + T^2_{22}\boldsymbol{e_2} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_2}$$

$$+ T^{222}\boldsymbol{e_2} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_2} = \left[\begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix}\right]$$

$$= (1)\left[\begin{bmatrix} 0.125000 & 0.216506 \\ 0.216506 & 0.375000 \end{bmatrix} \begin{bmatrix} 0.216506 & 0.375000 \\ 0.375000 & 0.649519 \end{bmatrix}\right]$$

$$+(2)\left[\begin{bmatrix} -0.433013 & 0.250000 \\ -0.750000 & 0.433013 \end{bmatrix} \begin{bmatrix} -0.750000 & 0.433013 \\ -1.299038 & 0.750000 \end{bmatrix}\right]$$

$$+(3)\left[\begin{bmatrix} -0.108253 & -0.187500 \\ 0.062500 & 0.108253 \end{bmatrix} \begin{bmatrix} -0.187500 & -0.324760 \\ 0.108253 & 0.187500 \end{bmatrix}\right]$$

$$+(4)\left[\begin{bmatrix} 0.375000 & -0.216506 \\ -0.216506 & 0.125000 \end{bmatrix} \begin{bmatrix} 0.649519 & -0.375000 \\ -0.375000 & 0.216506 \end{bmatrix}\right]$$

$$+(5)\left[\begin{bmatrix} -0.433013 & -0.750000 \\ -0.750000 & -1.299038 \end{bmatrix} \begin{bmatrix} 0.250000 & 0.433013 \\ 0.433013 & 0.750000 \end{bmatrix}\right]$$

$$+(6)\left[\begin{bmatrix} 1.500000 & -0.866025 \\ 2.598076 & -1.500000 \end{bmatrix} \begin{bmatrix} -0.866025 & 0.500000 \\ -1.500000 & 0.866025 \end{bmatrix}\right]$$

$$+(7)\left[\begin{bmatrix} 0.375000 & 0.649519 \\ -0.216506 & -0.375000 \end{bmatrix} \begin{bmatrix} -0.216506 & -0.375000 \\ 0.125000 & 0.216506 \end{bmatrix}\right]$$

$$+(8)\left[\begin{bmatrix} -1.299038 & 0.750000 \\ 0.750000 & -0.433013 \end{bmatrix} \begin{bmatrix} 0.750000 & -0.433013 \\ -0.433013 & 0.250000 \end{bmatrix}\right]$$

$$= \left[\begin{bmatrix} -0.498153 & 0.888462 \\ 14.360894 & -19.518507 \end{bmatrix} \begin{bmatrix} 1.290386 & -2.157291 \\ -12.822355 & 16.039741 \end{bmatrix}\right] \tag{A.1}$$

Back to Example

Appendix B -Fourth Order Tensor Computed Using the Outer Product

Equation (125) ⇒

$$T^{il}_{jk}\boldsymbol{e_i}\otimes\boldsymbol{\omega^j}\otimes\boldsymbol{\omega^k}\otimes\boldsymbol{e_l} = (1)\begin{bmatrix}\begin{bmatrix}0.062500 & 0.108253\\0.108253 & 0.187500\end{bmatrix} & \begin{bmatrix}0.108253 & 0.187500\\0.187500 & 0.324760\end{bmatrix}\\\begin{bmatrix}0.108253 & 0.187500\\0.187500 & 0.324760\end{bmatrix} & \begin{bmatrix}0.187500 & 0.324760\\0.324760 & 0.562500\end{bmatrix}\end{bmatrix}$$

$$+\ (2)\begin{bmatrix}\begin{bmatrix}-0.216506 & 0.125000\\-0.375000 & 0.216506\end{bmatrix} & \begin{bmatrix}-0.375000 & 0.216506\\-0.649519 & 0.375000\end{bmatrix}\\\begin{bmatrix}-0.375000 & 0.216506\\-0.649519 & 0.375000\end{bmatrix} & \begin{bmatrix}-0.649519 & 0.375000\\-1.125000 & 0.649519\end{bmatrix}\end{bmatrix}$$

$$+\ (3)\begin{bmatrix}\begin{bmatrix}-0.054127 & -0.093750\\0.031250 & 0.054127\end{bmatrix} & \begin{bmatrix}-0.093750 & -0.162380\\0.054127 & 0.093750\end{bmatrix}\\\begin{bmatrix}-0.093750 & -0.162380\\0.054127 & 0.093750\end{bmatrix} & \begin{bmatrix}-0.162380 & -0.281250\\0.093750 & 0.162380\end{bmatrix}\end{bmatrix}$$

$$+\ (4)\begin{bmatrix}\begin{bmatrix}0.187500 & -0.108253\\-0.108253 & 0.062500\end{bmatrix} & \begin{bmatrix}0.324760 & -0.187500\\-0.187500 & 0.108253\end{bmatrix}\\\begin{bmatrix}0.324760 & -0.187500\\-0.187500 & 0.108253\end{bmatrix} & \begin{bmatrix}0.562500 & -0.324760\\-0.324760 & 0.187500\end{bmatrix}\end{bmatrix}$$

$$+\ (5)\begin{bmatrix}\begin{bmatrix}-0.054127 & -0.093750\\-0.093750 & -0.162380\end{bmatrix} & \begin{bmatrix}0.031250 & 0.054127\\0.054127 & 0.093750\end{bmatrix}\\\begin{bmatrix}-0.093750 & -0.162380\\-0.162380 & -0.281250\end{bmatrix} & \begin{bmatrix}0.054127 & 0.093750\\0.093750 & 0.162380\end{bmatrix}\end{bmatrix}$$

$$+(6)\begin{bmatrix}\begin{bmatrix}0.187500 & -0.108253\\0.324760 & -0.187500\end{bmatrix} & \begin{bmatrix}-0.108253 & 0.062500\\-0.187500 & 0.108253\end{bmatrix}\\\begin{bmatrix}0.324760 & -0.187500\\0.562500 & -0.324760\end{bmatrix} & \begin{bmatrix}-0.187500 & 0.108253\\-0.324760 & 0.187500\end{bmatrix}\end{bmatrix}$$

$$+\ (7)\begin{bmatrix}\begin{bmatrix}0.046875 & 0.081190\\-0.027063 & -0.046875\end{bmatrix} & \begin{bmatrix}-0.027063 & -0.046875\\0.015625 & 0.027063\end{bmatrix}\\\begin{bmatrix}0.081190 & 0.140625\\-0.046875 & -0.081190\end{bmatrix} & \begin{bmatrix}-0.046875 & -0.081190\\0.027063 & 0.046875\end{bmatrix}\end{bmatrix}$$

$$+(8)\begin{bmatrix}\begin{bmatrix}-0.162380 & 0.093750\\0.093750 & -0.054127\end{bmatrix} & \begin{bmatrix}0.093750 & -0.054127\\-0.054127 & 0.031250\end{bmatrix}\\\begin{bmatrix}-0.281250 & 0.162380\\0.162380 & -0.093750\end{bmatrix} & \begin{bmatrix}0.162380 & -0.093750\\-0.093750 & 0.054127\end{bmatrix}\end{bmatrix}$$

$$+\ (9)\begin{bmatrix}\begin{bmatrix}-0.216506 & -0.375000\\-0.375000 & -0.649519\end{bmatrix} & \begin{bmatrix}-0.375000 & -0.649519\\-0.649519 & -1.125000\end{bmatrix}\\\begin{bmatrix}0.125000 & 0.216506\\0.216506 & 0.375000\end{bmatrix} & \begin{bmatrix}0.216506 & 0.375000\\0.375000 & 0.649519\end{bmatrix}\end{bmatrix}$$

$$+ (10)\begin{bmatrix}\begin{bmatrix}0.750000 & -0.433013\\1.299038 & -0.750000\end{bmatrix} & \begin{bmatrix}1.299038 & -0.750000\\2.250000 & -1.299038\end{bmatrix}\\\begin{bmatrix}-0.433013 & 0.250000\\-0.750000 & 0.433013\end{bmatrix} & \begin{bmatrix}-0.750000 & 0.433013\\-1.299038 & 0.750000\end{bmatrix}\end{bmatrix}$$

$$+ (11)\begin{bmatrix}\begin{bmatrix}0.187500 & 0.324760\\-0.108253 & -0.187500\end{bmatrix} & \begin{bmatrix}0.324760 & 0.562500\\-0.187500 & -0.324760\end{bmatrix}\\\begin{bmatrix}-0.108253 & -0.187500\\0.062500 & 0.108253\end{bmatrix} & \begin{bmatrix}-0.187500 & -0.324760\\0.108253 & 0.187500\end{bmatrix}\end{bmatrix}$$

$$+(12)\begin{bmatrix}\begin{bmatrix}-0.649519 & 0.375000\\0.375000 & -0.216506\end{bmatrix} & \begin{bmatrix}-1.125000 & 0.649519\\0.649519 & -0.375000\end{bmatrix}\\\begin{bmatrix}0.375000 & -0.216506\\-0.216506 & 0.125000\end{bmatrix} & \begin{bmatrix}0.649519 & -0.375000\\-0.375000 & 0.216506\end{bmatrix}\end{bmatrix}$$

$$+ (13)\begin{bmatrix}\begin{bmatrix}0.187500 & 0.324760\\0.324760 & 0.562500\end{bmatrix} & \begin{bmatrix}-0.108253 & -0.187500\\-0.187500 & -0.324760\end{bmatrix}\\\begin{bmatrix}-0.108253 & -0.187500\\-0.187500 & -0.324760\end{bmatrix} & \begin{bmatrix}0.062500 & 0.108253\\0.108253 & 0.187500\end{bmatrix}\end{bmatrix}$$

$$+ (14)\begin{bmatrix}\begin{bmatrix}-0.649519 & 0.375000\\-1.125000 & 0.649519\end{bmatrix} & \begin{bmatrix}0.375000 & -0.216506\\0.649519 & -0.375000\end{bmatrix}\\\begin{bmatrix}0.375000 & -0.216506\\0.649519 & -0.375000\end{bmatrix} & \begin{bmatrix}-0.216506 & 0.125000\\-0.375000 & 0.216506\end{bmatrix}\end{bmatrix}$$

$$+ (15)\begin{bmatrix}\begin{bmatrix}-0.162380 & -0.281250\\0.093750 & 0.162380\end{bmatrix} & \begin{bmatrix}0.093750 & 0.162380\\-0.054127 & -0.093750\end{bmatrix}\\\begin{bmatrix}0.093750 & 0.162380\\-0.054127 & -0.093750\end{bmatrix} & \begin{bmatrix}-0.054127 & -0.093750\\0.031250 & 0.054127\end{bmatrix}\end{bmatrix}$$

$$+ (16)\begin{bmatrix}\begin{bmatrix}0.562500 & -0.324760\\-0.324760 & 0.187500\end{bmatrix} & \begin{bmatrix}-0.324760 & 0.187500\\0.187500 & -0.108253\end{bmatrix}\\\begin{bmatrix}-0.324760 & 0.187500\\0.187500 & -0.108253\end{bmatrix} & \begin{bmatrix}0.187500 & -0.108253\\-0.108253 & 0.062500\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}\begin{bmatrix}-0.171187 & 0.268250\\-1.334078 & 2.170071\end{bmatrix} & \begin{bmatrix}0.183871 & -0.128967\\28.352762 & -40.451176\end{bmatrix}\\\begin{bmatrix}0.331072 & -0.515024\\3.216705 & -5.098377\end{bmatrix} & \begin{bmatrix}0.917667 & -1.616326\\-23.013998 & 31.083449\end{bmatrix}\end{bmatrix} \tag{B.1}$$

To illustrate the calculation better, we will show how to evaluate one of the fourth order basis.

To evaluate

$$\boldsymbol{e_2} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{\omega^2} \otimes \boldsymbol{e_1} = [\boldsymbol{e_2} \otimes \boldsymbol{\omega^2}] \otimes [\boldsymbol{\omega^2} \otimes \boldsymbol{e_1}]$$

$\Rightarrow$

$$\boldsymbol{e_2} \otimes \boldsymbol{\omega^2} = \begin{bmatrix}-1.73205081\\1\end{bmatrix}\begin{bmatrix}-0.433013 & 0.250000\end{bmatrix} = \begin{bmatrix}0.7500006 & -0.43301275\\-0.433013 & 0.25\end{bmatrix} \tag{B.2}$$

$$\boldsymbol{\omega}^2 \otimes \boldsymbol{e}_1 = \begin{bmatrix} -0.433013 \\ 0.250000 \end{bmatrix} \begin{bmatrix} 0.5 & 0.8660254 \end{bmatrix} = \begin{bmatrix} -0.2165065 & -0.37500008 \\ 0.125 & 0.21650625 \end{bmatrix} \tag{B.3}$$

Using the equation of multiplying each element of $[\boldsymbol{e}_2 \otimes \boldsymbol{\omega}^2]$ by $[\boldsymbol{\omega}^2 \otimes \boldsymbol{e}_1]$

$$[\boldsymbol{e}_2 \otimes \boldsymbol{\omega}^2] \otimes [\boldsymbol{\omega}^2 \otimes \boldsymbol{e}_1]$$

$$= \begin{bmatrix} 0.7500006 \begin{bmatrix} -0.2165065 & -0.37500008 \\ 0.125 & 0.21650625 \end{bmatrix} & -0.43301275 \begin{bmatrix} -0.2165065 & -0.37500008 \\ 0.125 & 0.21650625 \end{bmatrix} \\ -0.433013 \begin{bmatrix} -0.2165065 & -0.37500008 \\ 0.125 & 0.21650625 \end{bmatrix} & 0.25 \begin{bmatrix} -0.2165065 & -0.37500008 \\ 0.125 & 0.21650625 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} -0.162380 & -0.281250 \\ 0.093750 & 0.162380 \end{bmatrix} & \begin{bmatrix} 0.093750 & 0.162380 \\ -0.054127 & -0.093750 \end{bmatrix} \\ \begin{bmatrix} 0.093750 & 0.162380 \\ -0.054127 & -0.093750 \end{bmatrix} & \begin{bmatrix} -0.054127 & -0.093750 \\ 0.031250 & 0.054127 \end{bmatrix} \end{bmatrix} \tag{B.4}$$

Notice that equation (B.4) differs from the $\boldsymbol{e}_2 \otimes \boldsymbol{\omega}^2 \otimes \boldsymbol{\omega}^2 \otimes \boldsymbol{e}_1$ term in equation (B.1). This is the same as was seen for the case of the Third Order. Both terms are the same, but because of the order of the computations, the positions in the basis differ.

$$\begin{bmatrix} \begin{bmatrix} -0.162380 & 0.093750 \\ 0.093750 & -0.054127 \end{bmatrix} & \begin{bmatrix} 0.093750 & -0.054127 \\ -0.054127 & 0.031250 \end{bmatrix} \\ \begin{bmatrix} -0.281250 & 0.162380 \\ 0.162380 & -0.093750 \end{bmatrix} & \begin{bmatrix} 0.162380 & -0.093750 \\ -0.093750 & 0.054127 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} -0.162380 & -0.281250 \\ 0.093750 & 0.162380 \end{bmatrix} & \begin{bmatrix} 0.093750 & 0.162380 \\ -0.054127 & -0.093750 \end{bmatrix} \\ \begin{bmatrix} 0.093750 & 0.162380 \\ -0.054127 & -0.093750 \end{bmatrix} & \begin{bmatrix} -0.054127 & -0.093750 \\ 0.031250 & 0.054127 \end{bmatrix} \end{bmatrix}$$

Back to Example