Coordinate Transform of Christoffel Symbols of the Second Kind
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Introduction

Equation (1) shows the definition for the Christoffel of the Second Kind in index form.

$$\frac{\partial \boldsymbol{e_i}}{\partial q^j} = \Gamma_{ij}^k \boldsymbol{e_k} \tag{1}$$

Equation (2) shows the matrix form for equation (1) from a previous writeup.[1]

$$\frac{\partial E_n}{\partial q^p} = \Gamma_p E_n \tag{2}$$

where

E_n are the basis vectors as row vectors

Equation (2) is valid in any coordinate system E_n with coordinates n and p. Note: n and p refer to the same coordinate system with different indices. In this write up, there will be a transformation involving multiple coordinate systems, so equation (2) includes the subscript n to keep track of the different coordinate systems. As will be discussed later in this write-up, the A and B matrix will be defined using subscripts that indicate the coordinates they are transforming between to make it easier to keep track of the various coordinate systems.

Use equation (3) to solve for Γ_p [2]

$$W_n = (E_n^T)^{-1} = (E_n^{-1})^T \Rightarrow$$

$$W_n^T = E_n^{-1} \Rightarrow \tag{3}$$

Using equation (3), equation (2) can be solved for $\Gamma_p \Rightarrow$

[1] Christoffel Symbols in Matrix Form
[2] Coordinates Summary

$$\Gamma_p = \frac{\partial E_n}{\partial q^p} E_n^{-1} = \frac{\partial E_n}{\partial q^p} W_n^T \tag{4}$$

Note: that Γ_p in index form is Γ_{ip}^j

Equation (4) is a general formula because the index p can vary. To see this, consider the derivative of one vector with respect to another as shown in equation (5).

$$\frac{\partial u^j}{\partial v^j} = \begin{bmatrix} \frac{\partial u^1}{\partial v^1} & \cdots & \frac{\partial u^n}{\partial v^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^1}{\partial v^n} & \cdots & \frac{\partial u^n}{\partial v^n} \end{bmatrix} \tag{5}$$

Now apply this same pattern to taking the derivative of a matrix with respect to a vector as shown in equation (6).

$$\frac{\partial E_n}{\partial q^p} = \begin{bmatrix} \frac{\partial E_n}{\partial q^1} \\ \vdots \\ \frac{\partial E_n}{\partial q^n} \end{bmatrix} \tag{6}$$

Equation (6) is a column vector where each element is a matrix.
Putting equation (2) in this format ⇒

$$\frac{\partial E_n}{\partial q^p} = \begin{bmatrix} \Gamma_1 E_n \\ \vdots \\ \Gamma_n E_n \end{bmatrix} \tag{7}$$

Using equation (3) to solve for Γ_p ⇒

$$\begin{bmatrix} \Gamma_1 E_n W_n^T \\ \vdots \\ \Gamma_n E_n W_n^T \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_n \end{bmatrix} = \frac{\partial E_n}{\partial q^p} W_n^T = \begin{bmatrix} \frac{\partial E_n}{\partial q^1} W_n^T \\ \vdots \\ \frac{\partial E_n}{\partial q^n} W_n^T \end{bmatrix} \tag{8}$$

Transformation of Christoffel Symbols of the Second Kind in Matrix Form

Start with the coordinate transform definition in equation (9).

$$\bar{E} = E_{\bar{n}} = AE_n = A_{\bar{n}n}E_n \tag{9}$$

where

$E_{\bar{n}}$ are the basis vectors in the primed coordinate system

$$A = \left[\frac{\partial \boldsymbol{q}}{\partial \bar{\boldsymbol{q}}}\right] = \frac{\partial q^n}{\partial q^{\bar{n}}} = \begin{bmatrix} \frac{\partial q^1}{\partial q^{\bar{1}}} & \cdots & \frac{\partial q^n}{\partial q^{\bar{1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial q^1}{\partial q^{\bar{n}}} & \cdots & \frac{\partial q^n}{\partial q^{\bar{n}}} \end{bmatrix} = A_{\bar{n}n}$$

Note that we are putting the bar over the indices - $q^{\bar{p}}$ and $\Gamma_{\bar{p}}$ to describe the objects in the primed coordinate system. This notation emphasizes that $\Gamma_{\bar{p}}$ and Γ_p are the same mathematical object in different coordinate systems. Note: saying something is the same mathematical object does not mean it necessarily transforms like a tensor. Also Note, the upper index represents the column index while the lower index represents the row index.

Now take the derivative of equation (9) with respect to $q^{\bar{p}}$.

$$\Gamma_{\bar{p}}E_{\bar{n}} = \frac{\partial E_{\bar{n}}}{\partial q^{\bar{p}}} = \frac{\partial q^l}{\partial q^{\bar{p}}}\frac{\partial (A_{\bar{n}n}E_n)}{\partial q^l} = \frac{\partial q^l}{\partial q^{\bar{p}}}\left[\frac{\partial A_{\bar{n}n}}{\partial q^l}E + A_{\bar{n}n}\frac{\partial E}{\partial q^l}\right] = \frac{\partial A_{\bar{n}n}}{\partial q^{\bar{p}}}E + \frac{\partial q^l}{\partial q^{\bar{p}}}A_{\bar{n}n}\frac{\partial E}{\partial q^l} \tag{10}$$

Notice that there is the following simplification in equation (10) using the chain rule:

$$\frac{\partial q^l}{\partial q^{\bar{p}}}\frac{\partial A_{\bar{n}n}}{\partial q^l} = \frac{\partial A_{\bar{n}n}}{\partial q^{\bar{p}}}$$

We could use equation (10) without the simplification if it better suites the situation.

From Coordinate Summary[3]

$$\bar{W} = W_{\bar{n}} = [W^T B]^T = B^T W = B^T_{n\bar{n}}W_n = B_{\bar{n}n}W_n \Rightarrow \tag{11}$$

[3] Coordinates Summary

where

$$B=\frac{\partial\bar{\boldsymbol{q}}}{\partial\boldsymbol{q}}=\begin{bmatrix}\frac{\partial q^{\bar{1}}}{\partial q^{1}} & \cdots & \frac{\partial q^{\bar{n}}}{\partial q^{1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial q^{\bar{1}}}{\partial q^{n}} & \cdots & \frac{\partial q^{\bar{n}}}{\partial q^{n}}\end{bmatrix}=B_{n\bar{n}}$$

W_n is a matrix of the unprimed reciprocal basis vectors as row vectors.

W_n can also be thought of as an arbitrary row - n^{th}row of the matrix W and then equation (11) implies Einstein's summation convention as shown below.

$$B_{\bar{n}n}W_n=\begin{bmatrix}B_{\bar{1}1}[W_1]+ & \cdots & +B_{\bar{1}n}[W_n]\\ \vdots & \ddots & \vdots\\ B_{\bar{n}1}[W_1]+ & \cdots & B_{\bar{n}n}[W_n]\end{bmatrix}$$

W_n is the n^{th} row vector of W, so $B_{\bar{n}n}W_n$ is a matrix.

Multiplying both sides of equation (10) by $\bar{W}^T$ and using equation (11) ⇒

$$\Gamma_{\bar{p}}\bar{E}\bar{W}^T=\Gamma_{\bar{p}}=\frac{\partial A}{\partial q^{\bar{p}}}E\bar{W}^T+\frac{\partial q^l}{\partial q^{\bar{p}}}A\frac{\partial E}{\partial q^l}\bar{W}^T=\frac{\partial A}{\partial q^{\bar{p}}}EW^TB+\frac{\partial q^l}{\partial q^{\bar{p}}}A\frac{\partial E}{\partial q^l}W^TB$$

$$=\frac{\partial A}{\partial q^{\bar{p}}}B+\frac{\partial q^l}{\partial q^{\bar{p}}}A\frac{\partial E}{\partial q^l}W^TB=\frac{\partial A}{\partial q^{\bar{p}}}B+\frac{\partial q^l}{\partial q^{\bar{p}}}A\Gamma_lEW^TB=\frac{\partial A}{\partial q^{\bar{p}}}B+\frac{\partial q^l}{\partial q^{\bar{p}}}A\Gamma_lB$$

$$=\frac{\partial A_{\bar{n}n}}{\partial q^{\bar{p}}}B_{n\bar{n}}+\frac{\partial q^l}{\partial q^{\bar{p}}}A_{\bar{n}n}\Gamma_lB_{n\bar{n}} \tag{12}$$

Equation (12) is the transformation equation for a Christoffel Symbol of the Second Kind in matrix form. Applying the same pattern as in equation (8), equation (12) ⇒

$$\Gamma_{\bar{p}}=\begin{bmatrix}\Gamma_{\bar{1}}\\ \vdots\\ \Gamma_{\bar{n}}\end{bmatrix}=\begin{bmatrix}\frac{\partial A_{\bar{n}n}}{\partial q^{\bar{1}}}B_{n\bar{n}}\\ \vdots\\ \frac{\partial A_{\bar{n}n}}{\partial q^{\bar{n}}}B_{n\bar{n}}\end{bmatrix}+\begin{bmatrix}A_{\bar{n}n}\frac{\partial q^l}{\partial q^{\bar{1}}}\Gamma_lB_{n\bar{n}}\\ \vdots\\ A_{\bar{n}n}\frac{\partial q^l}{\partial q^{\bar{n}}}\Gamma_lB_{n\bar{n}}\end{bmatrix} \tag{13}$$

Transformation of Christoffel Symbols in Index Form

Now, derive the transformation equations in index form This discussion follows the approach used in[4].

Equation (14) shows the Christoffel of the Second Kind in the primed coordinate system using index notation.

$$\frac{\partial \boldsymbol{e_{\bar{\imath}}}}{\partial q^{\bar{\jmath}}} = \Gamma^{\bar{k}}_{\bar{\imath}\,\bar{\jmath}} \boldsymbol{e_{\bar{k}}} \tag{14}$$

where

$$\boldsymbol{e_{\bar{\imath}}} = \frac{\partial q^m}{\partial q^{\bar{\imath}}} \boldsymbol{e_m}$$

From the chain rule

$$\frac{\partial}{\partial q^{\bar{\jmath}}} = \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial}{\partial q^l} \;\Rightarrow$$

$$\frac{\partial \boldsymbol{e_{\bar{\imath}}}}{\partial q^{\bar{\jmath}}} = \frac{\partial}{\partial q^{\bar{\jmath}}}\left(\frac{\partial q^m}{\partial q^{\bar{\imath}}} \boldsymbol{e_m}\right) = \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial}{\partial q^l}\left(\frac{\partial q^m}{\partial q^{\bar{\imath}}} \boldsymbol{e_m}\right)$$

$$= \frac{\partial q^l}{\partial q^{\bar{\jmath}}}\left[\frac{\partial^2 q^m}{\partial q^{\bar{\imath}} \partial q^l} \boldsymbol{e_m} + \frac{\partial q^m}{\partial q^{\bar{\imath}}} \frac{\partial \boldsymbol{e_m}}{\partial q^l}\right] = \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial^2 q^m}{\partial q^{\bar{\imath}} \partial q^l} \boldsymbol{e_m} + \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial q^m}{\partial q^{\bar{\imath}}} \Gamma^n_{ml} \boldsymbol{e_n} \tag{15}$$

Interchange m and n in the second term of equation (15) so that the basis vectors $\boldsymbol{e_k}$ have the same index.

$$\frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial^2 q^m}{\partial q^{\bar{\imath}} \partial q^l} \boldsymbol{e_m} + \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial q^n}{\partial q^{\bar{\imath}}} \Gamma^m_{nl} \boldsymbol{e_m} = \left[\frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial^2 q^m}{\partial q^{\bar{\imath}} \partial q^l} + \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial q^n}{\partial q^{\bar{\imath}}} \Gamma^m_{nl}\right] \boldsymbol{e_m} \tag{16}$$

Setting equation (14) equal to equation (16) ⇒

$$\Gamma^{\bar{k}}_{\bar{\imath}\,\bar{\jmath}} \boldsymbol{e_{\bar{k}}} = \Gamma^{\bar{k}}_{\bar{\imath}\,\bar{\jmath}} \frac{\partial q^m}{\partial q^{\bar{k}}} \boldsymbol{e_m} = \left[\frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial^2 q^m}{\partial q^{\bar{\imath}} \partial q^l} + \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial q^n}{\partial q^{\bar{\imath}}} \Gamma^m_{nl}\right] \boldsymbol{e_m} \tag{17}$$

Equation (17) ⇒

[4] David McMahon, Relativity Demystified, McGraw-Hill, 2006, pp. 68

$$\Gamma_{\bar{\iota}\bar{\jmath}}^{\bar{k}} \frac{\partial q^m}{\partial q^{\bar{k}}} = \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial^2 q^m}{\partial q^{\bar{\iota}} \partial q^l} + \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial q^n}{\partial q^{\bar{\iota}}} \Gamma_{nl}^m \Rightarrow$$

$$\Gamma_{\bar{\iota}\bar{\jmath}}^{\bar{k}} = \frac{\partial q^{\bar{k}}}{\partial q^m} \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial^2 q^m}{\partial q^{\bar{\iota}} \partial q^l} + \frac{\partial q^{\bar{k}}}{\partial q^m} \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial q^n}{\partial q^{\bar{\iota}}} \Gamma_{nl}^m \tag{18}$$

Equation (18) is the equation of how the Christoffel Symbol of the Second Kind transforms under a coordinate change. The transformation rule for a 3^{rd} order tensor is given by equation (19).

$$A_{\bar{n}\bar{l}}^{\bar{m}} = \frac{\partial q^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial q^{\bar{n}}} \frac{\partial q^l}{\partial q^{\bar{l}}} A_{nl}^m \tag{19}$$

The second part of equation (18) gives the correct form of a tensor transformation, but the first part does not; therefore, Γ_{nl}^m does not transform as a tensor.

Note: we can use the same simplification – using the chain rule - in equation (19) that was done in equation (10).

$$\frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial^2 q^m}{\partial q^{\bar{\iota}} \partial q^l} = \frac{\partial^2 q^m}{\partial q^{\bar{\iota}} \partial q^{\bar{\jmath}}} \tag{20}$$

Then equation (18) becomes

$$\Gamma_{\bar{\iota}\bar{\jmath}}^{\bar{k}} = \frac{\partial q^{\bar{k}}}{\partial q^m} \frac{\partial^2 q^m}{\partial q^{\bar{\iota}} \partial q^{\bar{\jmath}}} + \frac{\partial q^{\bar{k}}}{\partial q^m} \frac{\partial q^l}{\partial q^{\bar{\jmath}}} \frac{\partial q^n}{\partial q^{\bar{\iota}}} \Gamma_{nl}^m \tag{21}$$

Equivalence of the Matrix Form with the Index Form

To show that the matrix form of the equations is the same as the index form, try to convert the matrix form to the index form and see if the index versions of the equations are the same. Start with the first part of equation (12) ⇒

$$A_{\bar{\iota}j} = \frac{\partial q^j}{\partial q^{\bar{\iota}}} \tag{22}$$

From the definition in equation (4), the upper index in (22) changes with column and the lower index changes with row. Convert the first term in equation (12) ⇒

$$\frac{\partial A_{\bar{\iota}j}}{\partial q^{\bar{p}}} = \frac{\partial^2 q^j}{\partial q^{\bar{\iota}} \partial q^{\bar{p}}} = \left[\frac{\partial A}{\partial q^{\bar{p}}}\right]_{\bar{\iota}j} \tag{23}$$

$$B_{j\bar{\imath}} = \frac{\partial q^{\bar{\imath}}}{\partial q^{j}}$$

(24)

Using the definition of matrix multiplication ⇒

$$\frac{\partial A}{\partial \bar{q}^{p}} B = \frac{\partial A_{\bar{\imath}k}}{\partial q^{\bar{p}}} B_{k\bar{\jmath}} = \frac{\partial^2 q^k}{\partial q^{\bar{\imath}} q^{\bar{p}}} \frac{\partial q^{\bar{\jmath}}}{\partial q^k}$$

(25)

Simplify the second part of equation (12) ⇒

$$\frac{\partial q^l}{\partial q^{\bar{p}}} \Gamma_l = \frac{\partial q^l}{\partial q^{\bar{p}}} \Gamma_{il}^{j} \Rightarrow$$

$$A \frac{\partial q^l}{\partial q^{\bar{p}}} \Gamma_l B = A \frac{\partial q^l}{\partial q^{\bar{p}}} \Gamma_{il}^{j} B$$

(26)

Let

$$C_{ij} = \frac{\partial q^l}{\partial q^{\bar{p}}} \Gamma_{il}^{j}$$

(27)

Using equation (27) in equation (26) ⇒

$$CB = C_{kl} B_{lj} \Rightarrow$$

$$ACB = A_{ik} [CB]_{kj} = A_{ik} C_{km} B_{m\bar{\jmath}} = \frac{\partial q^k}{\partial q^{\bar{\imath}}} \frac{\partial q^l}{\partial q^{\bar{p}}} \Gamma_{kl}^{m} \frac{\partial q^{\bar{\jmath}}}{\partial q^m}$$

(28)

Putting equations (28) and (25) into equation (12) ⇒

$$\Gamma_{\bar{p}} = \Gamma_{\bar{\imath}\bar{p}}^{\bar{\jmath}} = \frac{\partial A_{\bar{n}n}}{\partial q^{\bar{p}}} B_{n\bar{n}} + \frac{\partial q^l}{\partial q^{\bar{p}}} A_{\bar{n}n} \Gamma_l B_{n\bar{n}} = \frac{\partial^2 q^k}{\partial q^{\bar{\imath}} q^{\bar{p}}} \frac{\partial q^{\bar{\jmath}}}{\partial q^k} + \frac{\partial q^k}{\partial q^{\bar{\imath}}} \frac{\partial q^l}{\partial q^{\bar{p}}} \Gamma_{kl}^{m} \frac{\partial q^{\bar{\jmath}}}{\partial q^m}$$

$$= \frac{\partial^2 q^k}{\partial q^{\bar{\imath}} q^{\bar{p}}} \frac{\partial q^{\bar{\jmath}}}{\partial q^k} + \frac{\partial q^{\bar{\jmath}}}{\partial q^m} \frac{\partial q^l}{\partial q^{\bar{p}}} \frac{\partial q^k}{\partial q^{\bar{\imath}}} \Gamma_{kl}^{m}$$

(29)

The terms of the second term in equation (29) have been rearranged so it can be more easily compared to equation (21). The Einstein summation convention correctly maps the terms. Equation (29) is the index version generated from the matrix version of the equations. Now compare equation (29) to equation (21). Table 1 shows the mappings of the two versions of the equations.

	Matrix Version	Index Version
First Term	$\Gamma^{\bar{j}}_{\bar{i}\bar{p}} \longrightarrow \frac{\partial^2 q^k}{\partial q^{\bar{i}} q^{\bar{p}}} \frac{\partial q^{\bar{j}}}{\partial q^k}$	$\Gamma^{\bar{k}}_{\bar{i}\,\bar{j}} \longrightarrow \frac{\partial q^{\bar{k}}}{\partial q^m} \frac{\partial^2 q^m}{\partial q^{\bar{i}} \partial q^{\bar{j}}}$
Second Term	$\Gamma^{\bar{j}}_{\bar{i}\bar{p}} \longrightarrow \frac{\partial q^{\bar{j}}}{\partial q^m} \frac{\partial q^l}{\partial q^{\bar{p}}} \frac{\partial q^k}{\partial q^{\bar{i}}} \Gamma^m_{kl}$	$\Gamma^{\bar{k}}_{\bar{i}\,\bar{j}} \longrightarrow \frac{\partial q^{\bar{k}}}{\partial q^m} \frac{\partial q^l}{\partial q^{\bar{j}}} \frac{\partial q^n}{\partial q^{\bar{i}}} \Gamma^m_{nl}$

Table 1

The indices in Table 1 map the same way for each version, so the matrix and index versions of the equations are equivalent. Equation (29) can give the same form as equation (18) if we plug equation (20) into it.

Transformation Example

Figure 1 shows a block diagram for this example.

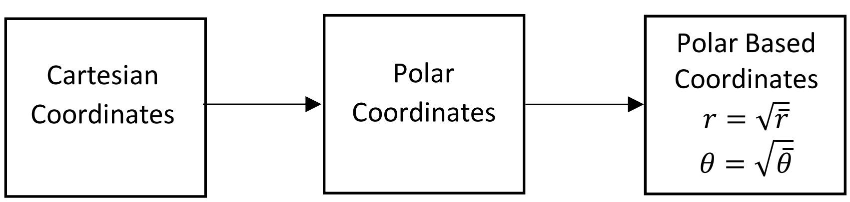

Figure 1

In this example, we do two calculations for the transformation of the Christoffel Symbols of the Second Kind. One from cartesian coordinates to polar coordinates and then polar to the polar based coordinate system using equation (2). The second one directly from polar to the polar based coordinate system using equation (12). Both computations get the same result as expected.

Let

$n = (x, y)$

$\bar{n} = (r, \theta)$

$\bar{\bar{n}} = (\bar{r}, \bar{\theta})$

Cartesian to Polar

Start by going from cartesian to polar – using the notation of equations (22) and (24) ⇒

$$A_{\bar{n}n} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -r\ sin(\theta) & r\ cos(\theta) \end{bmatrix} \tag{30}$$

$$B_{n\bar{n}} = \begin{bmatrix} cos(\theta) & -\frac{sin(\theta)}{r} \\ sin(\theta) & \frac{cos(\theta)}{r} \end{bmatrix} \tag{31}$$

$$E_n = I$$

$$E_{\bar{n}} = A_{\bar{n}n} E_n = A_{\bar{n}n} \tag{32}$$

$$\frac{\partial E_{\bar{n}}}{\partial r} = \frac{\partial A_{\bar{n}n}}{\partial r} = \frac{\partial}{\partial r}\begin{bmatrix} cos(\theta) & sin(\theta) \\ -r\ sin(\theta) & r\ cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -sin(\theta) & cos(\theta) \end{bmatrix} \tag{33}$$

Now use equation (12)

$$\Gamma_x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Gamma_y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Gamma_r = \frac{\partial A_{\bar{n}n}}{\partial r} B_{n\bar{n}} + \frac{\partial q^l}{\partial r} A_{\bar{n}n} \Gamma_l B_{n\bar{n}} = \frac{\partial A_{\bar{n}n}}{\partial r} B_{n\bar{n}} + \frac{\partial x}{\partial r} A_{\bar{n}n} \Gamma_x B_{n\bar{n}} + \frac{\partial y}{\partial r} A_{\bar{n}n} \Gamma_y B_{n\bar{n}} = \frac{\partial A_{\bar{n}n}}{\partial r} B_{n\bar{n}}$$

$$= \begin{bmatrix} 0 & 0 \\ -sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\theta) & -\frac{sin(\theta)}{r} \\ sin(\theta) & \frac{cos(\theta)}{r} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \tag{34}$$

Equation (34) gives the correct result.

$$\Gamma_\theta = \frac{\partial A_{\bar{n}n}}{\partial \theta} B_{n\bar{n}} + \frac{\partial q^l}{\partial \theta} A_{\bar{n}n} \Gamma_l B_{n\bar{n}} = \frac{\partial A_{\bar{n}n}}{\partial \theta} B_{n\bar{n}} + \frac{\partial x}{\partial \theta} A_{\bar{n}n} \Gamma_x B_{n\bar{n}} + \frac{\partial y}{\partial \theta} A_{\bar{n}n} \Gamma_y B_{n\bar{n}} = \frac{\partial A_{\bar{n}n}}{\partial \theta} B_{n\bar{n}}$$

$$= \frac{\partial}{\partial \theta} \begin{bmatrix} cos(\theta) & sin(\theta) \\ -r\ sin(\theta) & r\ cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\theta) & -\frac{sin(\theta)}{r} \\ sin(\theta) & \frac{cos(\theta)}{r} \end{bmatrix}$$

$$= \begin{bmatrix} -sin(\theta) & cos(\theta) \\ -r\ cos(\theta) & -r\ sin(\theta) \end{bmatrix} \begin{bmatrix} cos(\theta) & -\frac{sin(\theta)}{r} \\ sin(\theta) & \frac{cos(\theta)}{r} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{r} \\ -r & 0 \end{bmatrix} \tag{35}$$

Equation (35) gives the correct result.

Polar Based Transform

Equation (36) gives the equations for a polar based coordinate system.

$$r = \sqrt{\bar{r}}$$
$$\theta = \sqrt{\bar{\theta}} \tag{36}$$

Equation (37) show the coordinate transform equations from polar to the $(\bar{r} \quad \bar{\theta})$ coordinate system.

$$\frac{\partial}{\partial \bar{r}} = \frac{\partial r}{\partial \bar{r}} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \bar{r}} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial \bar{\theta}} = \frac{\partial r}{\partial \bar{\theta}} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \tag{37}$$

Equations (38) and (39) show the transformation matrices.

$$A_{\bar{\bar{n}}\bar{n}} = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix}$$

(38)

$$B_{\bar{n}\bar{\bar{n}}} = B_{\bar{\bar{n}}\bar{n}} = \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix}$$

(39)

Now go from cartesian to $(\bar{r} \quad \bar{\theta})$ coordinates - $n \to \bar{\bar{n}}$

$$A_{\bar{\bar{n}}n} = A_{\bar{\bar{n}}\bar{n}}A_{\bar{n}n} = \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \begin{bmatrix} cos(\theta) & sin(\theta) \\ -r\,sin(\theta) & r\,cos(\theta) \end{bmatrix} = \begin{bmatrix} \frac{cos(\theta)}{2\sqrt{\bar{r}}} & \frac{sin(\theta)}{2\sqrt{\bar{r}}} \\ -\frac{r\,sin(\theta)}{2\sqrt{\bar{\theta}}} & \frac{r\,cos(\theta)}{2\sqrt{\bar{\theta}}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} & \frac{sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} \\ -\frac{\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} & \frac{\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} \end{bmatrix}$$

(40)

$$B_{n\bar{\bar{n}}} = B_{n\bar{n}}B_{\bar{n}\bar{\bar{n}}} = \begin{bmatrix} cos(\theta) & -\frac{sin(\theta)}{r} \\ sin(\theta) & \frac{cos(\theta)}{r} \end{bmatrix} \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} 2\sqrt{\bar{r}}\,cos(\theta) & -\frac{2\sqrt{\bar{\theta}}\,sin(\theta)}{r} \\ 2\sqrt{\bar{r}}\,sin(\theta) & \frac{2\sqrt{\bar{\theta}}cos(\theta)}{r} \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right) & -\frac{2\sqrt{\bar{\theta}}\,sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \\ 2\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right) & \frac{2\sqrt{\bar{\theta}}cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \end{bmatrix}$$

(41)

Note that equation (41) has been set up so that matrix multiplication is valid. Another way of seeing this is $(A_2A_1)^{-1} = A_1^{-1}A_2^{-1} = A_{\bar{n}n}^{-1}A_{\bar{\bar{n}}\bar{n}}^{-1} = B_{n\bar{n}}B_{\bar{n}\bar{\bar{n}}}$ from properties of matrix inverses.

The transformation equations in matrix form are given by equations (42) and (43).

$\bar{E} = AE = E_{\bar{\bar{n}}} = A_{\bar{\bar{n}}n}E_n = A_{\bar{\bar{n}}n}$

(42)

where

$E_n = I$

Equation (43) gives the basis vector in the $(\bar{r} \quad \bar{\theta})$ ($\bar{\bar{n}}$ coordinate system).

$$E_{\bar{\bar{n}}} = \begin{bmatrix} \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} & \frac{sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} \\ -\frac{\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} & \frac{\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} \end{bmatrix}$$

(43)

Equation (44) computes the reciprocal basis vectors.

$$\bar{W} = B^T W = W_{\bar{\bar{n}}} = B^T_{\bar{n}\bar{\bar{n}}} W_{\bar{n}} = B_{\bar{\bar{n}}\bar{n}} W_{\bar{n}}$$

(44)

where

$W_{\bar{n}}$ is the polar coordinate reciprocal basis

$$W_{\bar{\bar{n}}} = B^T W = B^T_{\bar{n}\bar{\bar{n}}} W_{\bar{n}} = B_{\bar{\bar{n}}\bar{n}} W_{\bar{n}} = [B_{n\bar{\bar{n}}}]^T I = B_{\bar{\bar{n}}\bar{n}}$$

$$= \begin{bmatrix} 2\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right) & -\frac{2\sqrt{\bar{\theta}}\,sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \\ 2\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right) & \frac{2\sqrt{\bar{\theta}}cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \end{bmatrix}^T = \begin{bmatrix} 2\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\,\right) & 2\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\,\right) \\ -\frac{2\sqrt{\bar{\theta}}\,sin\left(\sqrt{\bar{\theta}}\,\right)}{\sqrt{\bar{r}}} & \frac{2\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\,\right)}{\sqrt{\bar{r}}} \end{bmatrix}$$

(45)

Computation of $\Gamma_{\bar{r}}$

Equation (46) gives the definition of $\Gamma_{\bar{r}}$ using equation (2).

$$\Gamma_{\bar{\bar{n}}} = \Gamma_{\bar{r}} = \frac{\partial \bar{E}}{\partial \bar{r}} \bar{W}^T = \frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}} W^T_{\bar{\bar{n}}} = \frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}} B_{n\bar{\bar{n}}}$$

(46)

where

$\bar{W}^T = W^T_{\bar{\bar{n}}} = [B^T W]^T = W^T B = W_n^T B_{n\bar{\bar{n}}} = B_{n\bar{\bar{n}}}$

$W_n^T = I$

$$\frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}} = \frac{\partial}{\partial \bar{r}} \begin{bmatrix} \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} & \frac{sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}} \\ -\frac{\sqrt{\bar{r}}\, sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} & \frac{\sqrt{\bar{r}}\, cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{47}$$

Computing equation (47) term by term

$$\left[\frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}}\right]_{11} = \frac{\partial}{\partial \bar{r}}\left(\frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}}\right) = \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2}\frac{\partial}{\partial \bar{r}}\left(\bar{r}^{\,-1/2}\right) = \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2}\left(-\frac{1}{2}\bar{r}^{\,-3/2}\right) = -\frac{cos\left(\sqrt{\bar{\theta}}\right)}{4r^{3/2}} \tag{48}$$

$$\left[\frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}}\right]_{12} = \frac{sin\left(\sqrt{\bar{\theta}}\right)}{2}\frac{\partial}{\partial \bar{r}}\left(\bar{r}^{\,-1/2}\right) = \left(\frac{sin\left(\sqrt{\bar{\theta}}\right)}{2}\right)\left(-\frac{1}{2}\bar{r}^{\,-3/2}\right) = -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\bar{r}^{3/2}} \tag{49}$$

$$\left[\frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}}\right]_{21} = -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}^{\,1/2}\right) = -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}}\left(\frac{1}{2}\bar{r}^{\,-1/2}\right) = -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}} \tag{50}$$

$$\left[\frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}}\right]_{22} = \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}^{\,1/2}\right) = \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}}\left(\frac{1}{2\sqrt{\bar{r}}}\right) = \frac{cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}} \tag{51}$$

Equations (48) – (51) ⇒

$$\frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}} = \begin{bmatrix} -\frac{cos\left(\sqrt{\bar{\theta}}\right)}{4r^{3/2}} & -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\bar{r}^{3/2}} \\ -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}} & \frac{\sqrt{\bar{r}}\, cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}} \end{bmatrix} \Rightarrow \tag{52}$$

Equation (46) ⇒

$$\Gamma_{\bar{r}} = \Gamma_{\bar{\bar{n}}} = \frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}} B_{n\bar{\bar{n}}} = \begin{bmatrix} -\frac{cos\left(\sqrt{\bar{\theta}}\right)}{4r^{3/2}} & -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\bar{r}^{3/2}} \\ -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}} & \frac{\sqrt{\bar{r}}\, cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}} \end{bmatrix} \begin{bmatrix} 2\sqrt{\bar{r}}\, cos\left(\sqrt{\bar{\theta}}\right) & -\frac{2\sqrt{\bar{\theta}} sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \\ 2\sqrt{\bar{r}}\, sin\left(\sqrt{\bar{\theta}}\right) & \frac{2\sqrt{\bar{\theta}} cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} \end{bmatrix}$$

(53)

Computing equation (53) term by term ⇒

$$[\Gamma_{\bar{r}}]_{11} = -\frac{cos\left(\sqrt{\bar{\theta}}\right)2\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{r}^{3/2}} - \frac{sin\left(\sqrt{\bar{\theta}}\right)2\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)}{4\bar{r}^{3/2}}$$

$$= -\frac{2\sqrt{\bar{r}}\left[cos^2\left(\sqrt{\bar{\theta}}\right) + sin^2\left(\sqrt{\bar{\theta}}\right)\right]}{4\bar{r}^{3/2}} = -\frac{1}{2\bar{r}}$$

(54)

$$[\Gamma_{\bar{r}}]_{12} = \frac{2\sqrt{\bar{\theta}}cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{r}^{3/2}}\frac{sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} - \frac{2\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right)}{4\bar{r}^{3/2}}\frac{cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} =$$

$$\frac{\sqrt{\bar{\theta}}\left[cos\left(\sqrt{\bar{\theta}}\right)sin\left(\sqrt{\bar{\theta}}\right) - sin\left(\sqrt{\bar{\theta}}\right)cos\left(\sqrt{\bar{\theta}}\right)\right]}{2\bar{r}^2} = 0$$

(55)

$$[\Gamma_{\bar{r}}]_{21} = -\frac{2\sqrt{\bar{r}}sin\left(\sqrt{\bar{\theta}}\right)\ cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}} + \frac{2\sqrt{\bar{r}}cos(\bar{\theta})\ sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}} = 0$$

(56)

$$\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}}\frac{2\sqrt{\bar{\theta}}\,sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}} + \frac{\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}}\frac{2\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}$$

$$[\Gamma_{\bar{r}}]_{22} = \left(-\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}}\right)\left(-\frac{2\sqrt{\bar{\theta}}\,sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\right) + \left(\frac{cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\theta}}\sqrt{\bar{r}}}\right)\left(\frac{2\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\right)$$

$$= \frac{2\sqrt{\bar{\theta}}}{4\sqrt{\bar{\theta}}\bar{r}}\left[sin^2\left(\sqrt{\bar{\theta}}\right) + cos^2\left(\sqrt{\bar{\theta}}\right)\right] = \frac{\sqrt{\bar{\theta}}}{2\sqrt{\bar{\theta}}} = \frac{1}{2\bar{r}}$$

(57)

Using equations (54) – (57) ⇒

$$\Gamma_{\bar{r}} = \begin{bmatrix} -\frac{1}{2\bar{r}} & 0 \\ 0 & \frac{1}{2\bar{r}} \end{bmatrix}$$

(58)

Now test equation (12) to see that we get the same result. For $\bar{n} \to \bar{\bar{n}}$ equation (12) ⇒

$$\Gamma_{\bar{\bar{n}}} = \frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial q^{\bar{p}}} B_{\bar{n}\bar{\bar{n}}} + \frac{\partial q^{l}}{\partial q^{\bar{p}}} A_{\bar{\bar{n}}\bar{n}} \Gamma_{l} B_{\bar{n}\bar{\bar{n}}} \Rightarrow \tag{12}$$

$$\Gamma_{\bar{r}} = \frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial \bar{r}} B_{\bar{n}\bar{\bar{n}}} + \frac{\partial r}{\partial \bar{r}} A_{\bar{\bar{n}}\bar{n}} \Gamma_{r} B_{\bar{n}\bar{\bar{n}}} + \frac{\partial \theta}{\partial \bar{r}} A_{\bar{\bar{n}}\bar{n}} \Gamma_{\theta} B_{\bar{n}\bar{\bar{n}}} = \frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial \bar{r}} B_{\bar{n}\bar{\bar{n}}} + A_{\bar{\bar{n}}\bar{n}} \left[\frac{\partial r}{\partial \bar{r}} \Gamma_{r} + \frac{\partial \theta}{\partial \bar{r}} \Gamma_{\theta}\right] B_{\bar{n}\bar{\bar{n}}} \tag{59}$$

The first term of equation (59) ⇒

$$\frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial \bar{r}} = \frac{\partial}{\partial \bar{r}} \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4\bar{r}^{3/2}} & 0 \\ 0 & 0 \end{bmatrix} \tag{60}$$

To evaluate the second terms of equation (59) ⇒

$$\frac{\partial r}{\partial \bar{r}} = \frac{1}{2\sqrt{\bar{r}}} \tag{61}$$

$$\frac{\partial \theta}{\partial \bar{r}} = 0 \tag{62}$$

Using the equations (61) and (62) in equation (59) ⇒

$$\Gamma_{\bar{r}} = \begin{bmatrix} -\frac{1}{4\bar{r}^{3/2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \left[\frac{1}{2\sqrt{\bar{r}}} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} + 0\right] \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2\bar{r}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2\sqrt{\bar{r}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2\sqrt{\bar{r}}r} \end{bmatrix} \begin{bmatrix} 2\sqrt{\bar{r}} & 0 \\ 0 & 2\sqrt{\bar{\theta}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2\bar{r}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2\sqrt{\bar{\bar{r}}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\theta}}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{\sqrt{\bar{\theta}}}{\sqrt{\bar{\bar{r}}}r} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\bar{r}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\bar{r}}}r} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2\bar{r}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2\sqrt{\bar{r}}\sqrt{\bar{\bar{r}}}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\bar{r}} & 0 \\ 0 & \frac{1}{2\bar{r}} \end{bmatrix} \tag{63}$$

Equation (63) is the same as equation (58) as expected.

Computation of $\Gamma_{\bar{\theta}}$

Equation (64) gives the definition of $\Gamma_{\bar{\theta}}$ using equation (2).

$$\Gamma_{\bar{\bar{n}}} = \Gamma_{\bar{\theta}} = \frac{\partial \bar{E}}{\partial \bar{\theta}} \bar{W}^T = \frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{\theta}} W_{\bar{\bar{n}}}^T = \frac{\partial E_{\bar{\bar{n}}}}{\partial \bar{r}} B_{n\bar{\bar{n}}} \tag{64}$$

$$\frac{\partial E_{\bar{\bar{n}}}}{\partial \theta} = \frac{\partial}{\partial \bar{\theta}} \begin{bmatrix} \frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\bar{r}}}} & \frac{sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\bar{r}}}} \\ -\frac{\sqrt{\bar{\bar{r}}}\, sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} & \frac{\sqrt{\bar{\bar{r}}}\, cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} \end{bmatrix} \tag{65}$$

Computing equation (65) term by term ⇒

$$\left[\frac{\partial E_{\bar{\bar{n}}}}{\partial \theta}\right]_{11} = \frac{\partial}{\partial \bar{\theta}} \left(\frac{cos\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\bar{r}}}} \right) = \frac{1}{2\sqrt{\bar{\bar{r}}}} \frac{\partial}{\partial \bar{\theta}} \left(cos\left(\sqrt{\bar{\theta}}\right)\right) = \frac{1}{2\sqrt{\bar{\bar{r}}}} \left(-sin\left(\sqrt{\bar{\theta}}\right)\right) \frac{1}{2} \bar{\theta}^{\,-1/2} = -\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\bar{r}}}\sqrt{\bar{\theta}}} \tag{66}$$

$$\left[\frac{\partial E_{\bar{\bar{n}}}}{\partial \theta}\right]_{12} = \frac{\partial}{\partial \bar{\theta}} \left(\frac{sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\bar{r}}}} \right) = \frac{1}{2\sqrt{\bar{\bar{r}}}} \frac{\partial}{\partial \bar{\theta}} \left(sin\left(\sqrt{\bar{\theta}}\right)\right) = \frac{1}{2\sqrt{\bar{\bar{r}}}} \left(cos\left(\sqrt{\bar{\theta}}\right)\right) \frac{1}{2} \bar{\theta}^{\,-1/2} = \frac{cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{\bar{r}}}\sqrt{\bar{\theta}}} \tag{67}$$

$$\left[\frac{\partial E_{\bar{\bar{p}}}}{\partial \theta}\right]_{21} = \frac{\partial}{\partial \bar{\theta}} \left(-\frac{\sqrt{\bar{\bar{r}}}\, sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{\theta}}} \right) = -\frac{\sqrt{\bar{\bar{r}}}}{2} \frac{\partial}{\partial \bar{\theta}} \left(\frac{sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{\theta}}} \right) =$$

$$-\frac{\sqrt{\bar{r}}}{2}\left(\frac{-\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)(1/2)\left(1/\sqrt{\bar{\theta}}\right)-sin\left(\sqrt{\bar{\theta}}\right)(1/2)\left(1/\sqrt{\bar{\theta}}\right)}{\bar{\theta}}\right)$$

$$=-\frac{\sqrt{\bar{r}}}{2}\left(\frac{\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)-sin\left(\sqrt{\bar{\theta}}\right)}{2\bar{\theta}^{\,3/2}}\right)=\frac{\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{r}}\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{\,3/2}} \tag{68}$$

$$\left[\frac{\partial E_{\bar{\bar{p}}}}{\partial\theta}\right]_{22}=\frac{\sqrt{\bar{r}}}{2}\frac{\partial}{\partial\bar{\theta}}\left(\frac{cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{\theta}}}\right)=\frac{\sqrt{\bar{r}}}{2}\left(\frac{-\sqrt{\bar{\theta}}\;sin\left(\sqrt{\bar{\theta}}\right)\left(1/\left(2\sqrt{\bar{\theta}}\right)\right)-cos\left(\sqrt{\bar{\theta}}\right)\left(1/\left(2\sqrt{\bar{\theta}}\right)\right)}{\bar{\theta}}\right)$$

$$=\frac{-\sqrt{\bar{\theta}}\sqrt{\bar{r}}sin\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}} \tag{69}$$

Using equations (66) – (69) ⇒

$$\frac{\partial E_{\bar{\bar{n}}}}{\partial\bar{\theta}}=\begin{bmatrix}-\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}} & \frac{cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}}\\ \frac{\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{r}}\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}} & \frac{-\sqrt{\bar{r}}\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}}\end{bmatrix} \tag{70}$$

$$\Gamma_{\bar{\theta}}=\frac{\partial E_{\bar{\bar{n}}}}{\partial\bar{\theta}}B_{n\bar{\bar{n}}}=\begin{bmatrix}-\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}} & \frac{cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}}\\ \frac{\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{r}}\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}} & \frac{-\sqrt{\bar{r}}\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}}\end{bmatrix}\cdot$$

$$\begin{bmatrix}2\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right) & -\frac{2\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\\ 2\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right) & \frac{2\sqrt{\bar{\theta}}cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\end{bmatrix} \tag{71}$$

Computing equation (71) term by term ⇒

$$[\Gamma_{\bar{\theta}}]_{11} = \left(-\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}}\right)2\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right) + \left(\frac{cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}}\right)\left(2\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)\right)$$

$$= \frac{-\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)cos\left(\sqrt{\bar{\theta}}\right) + \sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)sin\left(\sqrt{\bar{\theta}}\right)}{2\sqrt{\bar{r}}\sqrt{\bar{\theta}}} = 0$$

(72)

$$[\Gamma_{\bar{\theta}}]_{12} = \left(-\frac{sin\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}}\right)\left(-\frac{2\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\right) + \left(\frac{cos\left(\sqrt{\bar{\theta}}\right)}{4\sqrt{\bar{r}}\sqrt{\bar{\theta}}}\right)\left(\frac{2\sqrt{\bar{\theta}}cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\right)$$

$$= \frac{2\sqrt{\bar{\theta}}\left[sin^2\left(\sqrt{\bar{\theta}}\right) + cos^2\left(\sqrt{\bar{\theta}}\right)\right]}{4\bar{r}\sqrt{\bar{\theta}}} = \frac{2\sqrt{\bar{\theta}}}{4\bar{r}\sqrt{\bar{\theta}}} = \frac{1}{2\bar{r}}$$

(73)

$$[\Gamma_{\bar{\theta}}]_{21} = \left(\frac{\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right) - \sqrt{\bar{r}}\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}}\right)\left(2\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)\right)$$

$$+ \left(\frac{-\sqrt{\bar{r}}\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right) - \sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}}\right)\left(2\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right)\right)$$

$$= \frac{2\bar{r}\,sin\left(\sqrt{\bar{\theta}}\right)cos\left(\sqrt{\bar{\theta}}\right) - 2\bar{r}\sqrt{\bar{\theta}}cos^2(\bar{\theta}) - 2\sqrt{\bar{\theta}}\bar{r}\,sin^2\left(\sqrt{\bar{\theta}}\right) - 2\bar{r}\,cos\left(\sqrt{\bar{\theta}}\right)sin\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}}$$

$$= \frac{-2\bar{r}\left[\sqrt{\bar{\theta}}cos^2(\bar{\theta}) - \ sin\left(\sqrt{\bar{\theta}}\right)cos\left(\sqrt{\bar{\theta}}\right) + cos\left(\sqrt{\bar{\theta}}\right)sin\left(\sqrt{\bar{\theta}}\right) + \sqrt{\bar{\theta}}\,sin^2\left(\sqrt{\bar{\theta}}\right)\right]}{4\bar{\theta}^{3/2}}$$

$$= \frac{-2\bar{r}\left[\sqrt{\bar{\theta}}cos^2(\bar{\theta}) + \sqrt{\bar{\theta}}\,sin^2\left(\sqrt{\bar{\theta}}\right)\right]}{4\bar{\theta}^{3/2}} = -\frac{\bar{r}}{2\bar{\theta}}$$

(74)

$$[\Gamma_{\bar{\theta}}]_{22} = \left(\frac{\sqrt{\bar{r}}\,sin\left(\sqrt{\bar{\theta}}\right) - \sqrt{\bar{r}}\sqrt{\bar{\theta}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}}\right)\left(-\frac{2\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\right)$$

$$+\left(\frac{-\sqrt{\bar{r}}\sqrt{\bar{\theta}}sin\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{r}}\,cos\left(\sqrt{\bar{\theta}}\right)}{4\bar{\theta}^{3/2}}\right)\left(\frac{2\sqrt{\bar{\theta}}cos\left(\sqrt{\bar{\theta}}\right)}{\sqrt{\bar{r}}}\right)$$

$$=\frac{2\sqrt{\bar{r}}\left[-\sqrt{\theta}\,sin^2\left(\sqrt{\bar{\theta}}\right)+\theta sin\left(\sqrt{\bar{\theta}}\right)cos\left(\sqrt{\bar{\theta}}\right)-\ \theta\,sin\left(\sqrt{\bar{\theta}}\right)cos\left(\sqrt{\bar{\theta}}\right)-\sqrt{\bar{\theta}}cos^2\left(\sqrt{\bar{\theta}}\right)\right]}{\sqrt{\bar{r}}4\bar{\theta}^{3/2}}$$

$$=-\frac{2\sqrt{\bar{r}}\,\sqrt{\bar{\theta}}\left[\,sin^2\left(\sqrt{\bar{\theta}}\right)+cos^2\left(\sqrt{\bar{\theta}}\right)\right]}{\sqrt{\bar{r}}4\bar{\theta}^{3/2}}=-\frac{1}{2\bar{\theta}} \tag{75}$$

Putting equations (72) – (75) into a matrix ⇒

$$\Gamma_{\bar{\theta}}=\begin{bmatrix}0 & \frac{1}{2\bar{r}}\\ -\frac{\bar{r}}{2\bar{\theta}} & -\frac{1}{2\bar{\theta}}\end{bmatrix} \tag{76}$$

Now make the computation from equation (12)

$$\Gamma_{\bar{n}}=\frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial q^{\bar{p}}}B_{\bar{n}\bar{\bar{n}}}+\frac{\partial q^l}{\partial q^{\bar{p}}}A_{\bar{\bar{n}}\bar{n}}\Gamma_l B_{\bar{n}\bar{\bar{n}}}\Rightarrow$$

$$\Gamma_{\bar{\theta}}=\frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial\bar{\theta}}B_{\bar{n}\bar{\bar{n}}}+\frac{\partial r}{\partial\bar{\theta}}A_{\bar{\bar{n}}\bar{n}}\Gamma_r B_{\bar{n}\bar{\bar{n}}}+\frac{\partial\theta}{\partial\bar{\theta}}A_{\bar{\bar{n}}\bar{n}}\Gamma_\theta B_{\bar{n}\bar{\bar{n}}}=\frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial\bar{\theta}}B_{\bar{n}\bar{\bar{n}}}+A_{\bar{\bar{n}}\bar{n}}\left[\frac{\partial r}{\partial\bar{\theta}}\Gamma_r+\frac{\partial\theta}{\partial\bar{\theta}}\Gamma_\theta\right]B_{\bar{n}\bar{\bar{n}}} \tag{77}$$

Compute the first term in equation (77) ⇒

$$\frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial\bar{\theta}}=\frac{\partial}{\partial\bar{\theta}}\begin{bmatrix}\frac{1}{2\sqrt{\bar{r}}} & 0\\ 0 & \frac{1}{2\sqrt{\bar{\theta}}}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & -\frac{1}{4\bar{\theta}^{\ 3/2}}\end{bmatrix} \tag{78}$$

$$\frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial\bar{\theta}}B_{\bar{n}\bar{\bar{n}}}=\begin{bmatrix}0 & 0\\ 0 & -\frac{1}{4\bar{\theta}^{\ 3/2}}\end{bmatrix}\begin{bmatrix}2\sqrt{\bar{r}} & 0\\ 0 & 2\sqrt{\bar{\theta}}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & -\frac{1}{2\bar{\theta}}\end{bmatrix} \tag{79}$$

Compute the second terms in equation (77) ⇒

$$\frac{\partial r}{\partial \bar{\bar{\theta}}} = 0$$

$$\frac{\partial \theta}{\partial \bar{\bar{\theta}}} = \frac{1}{2\sqrt{\bar{\bar{\theta}}}} \Rightarrow$$

(80)

Using equation (80), equation (77) ⇒

$$\Gamma_{\bar{\bar{\theta}}} = \frac{\partial A_{\bar{\bar{n}}\bar{n}}}{\partial \bar{\bar{\theta}}} B_{\bar{n}\bar{\bar{n}}} + A_{\bar{\bar{n}}\bar{n}} \frac{1}{2\sqrt{\bar{\bar{\theta}}}} \Gamma_{\theta} B_{\bar{n}\bar{\bar{n}}} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2\bar{\bar{\theta}}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2\sqrt{\bar{\bar{r}}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\bar{\theta}}}} \end{bmatrix} \frac{1}{2\sqrt{\bar{\bar{\theta}}}} \begin{bmatrix} 0 & \frac{1}{r} \\ -r & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{\bar{\bar{r}}} & 0 \\ 0 & 2\sqrt{\bar{\bar{\theta}}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2\bar{\bar{\theta}}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2\sqrt{\bar{\bar{r}}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\bar{\theta}}}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2\sqrt{\bar{\bar{\theta}}}r} \\ -\frac{r}{2\sqrt{\bar{\bar{\theta}}}} & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{\bar{\bar{r}}} & 0 \\ 0 & 2\sqrt{\bar{\bar{\theta}}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2\bar{\bar{\theta}}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2\sqrt{\bar{\bar{r}}}} & 0 \\ 0 & \frac{1}{2\sqrt{\bar{\bar{\theta}}}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{r} \\ -\frac{r\sqrt{\bar{\bar{r}}}}{\sqrt{\bar{\bar{\theta}}}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2\bar{\bar{\theta}}} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2\sqrt{\bar{\bar{r}}}r} \\ -\frac{r\sqrt{\bar{\bar{r}}}}{2\bar{\bar{\theta}}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2\bar{\bar{\theta}}} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2\sqrt{\bar{\bar{r}}}\sqrt{\bar{\bar{r}}}} \\ -\frac{\sqrt{\bar{\bar{r}}}\sqrt{\bar{\bar{r}}}}{2\bar{\bar{\theta}}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2\bar{\bar{r}}} \\ -\frac{\bar{\bar{r}}}{2\bar{\bar{\theta}}} & -\frac{1}{2\bar{\bar{\theta}}} \end{bmatrix}$$

(81)

Equation (81) is the same as equation (76) as expected.